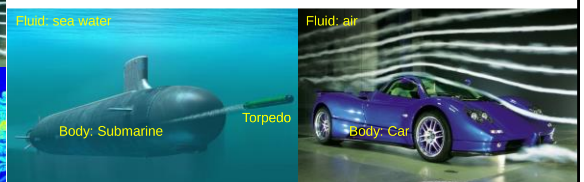
for 2nd grade undergraduate students

Ch.9 Flow over Immersed Bodies



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Ch. 9 Flow over Immersed Bodies

Learning Objectives in Ch.9

- ✓ identify and discuss the features of **external flow**.
- ✓ explain the fundamental characteristics of a **boundary layer**, including laminar, transitional, and turbulent regimes
- ✓ calculate **boundary layer parameters** for flow past a flat plate.
- ✓ provide a description of boundary layer **separation**.
- ✓ calculate the **lift** and **drag forces** for various objects.

S. Classification of fluid

$Re = \frac{\rho v D}{\mu} = \frac{Inertial\ force}{viscous\ force}$

$Ma = \frac{v}{a} = \frac{Fluid\ velocity}{Sound\ speed}$

- Fluids
  - Viscosity
    - Inviscid fluids (ideal fluids), $\mu = 0$ ⇨ Euler Equation
      - $\tau = \mu \frac{\partial u}{\partial y} = 0$
      - $\rho \frac{D\vec{V}}{Dt} = \rho \vec{f} - \Delta p$
    - Viscous fluids, $\mu \neq 0$
      - Newtonian ⇨ N-S Equation
        - $\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial \theta}{\partial t}$
        - $\rho \frac{D\vec{V}}{Dt} = \rho \vec{f} - \Delta p + \mu \nabla^2 \vec{V} + \frac{1}{3}\mu \nabla(\nabla \cdot \vec{V})$
      - Non-Newtonian ⇨ Rheology
        - $\tau \neq \mu \frac{\partial u}{\partial y}$
  - Compressibility
    - Incompressible fluids $(Ma < 0.3)$, $\rho = const.$
    - Compressible fluids, $\rho \neq const.$ ⇨ Gas dynamic $(Ma > 0.3)$

S. Scope of each chapter

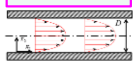
Laminar & Turbulent flow in internal viscous flow: Ch.8
Laminar & Turbulent flow in external viscous flow: Ch.9

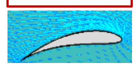
$Re = \frac{\rho v D}{\mu} = \frac{\text{Inertial force}}{\text{viscos force}}$

$Ma = \frac{v}{a} = \frac{\text{Fluid velocity}}{\text{Sound speed}}$

Continuum Mechanics (Fluid flow)

- Inviscid (μ=0) $\tau = \mu \frac{\partial u}{\partial y} = 0$
  - Incompressible: $\rho = const.$ (Ma < 0.3)
  - Compressible: $\rho \neq const.$ (Ma > 0.3)
- viscous($\mu \neq 0$) $\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial \theta}{\partial t}$
  - Laminar (Re < 2,100) — Hagen-Poiseuille's law: $Q = \frac{\pi d^4}{128 \mu l}(p_1 - p_2)$
  - Turbulent (Re ≥ 4,000) — Darcy-Weisbach's la: $h = \lambda \frac{l}{d}\frac{V^2}{2g}$, Moody diagram
  - Internal
  - External





S. Summary of internal flow in Ch.8

Internal flow(Pipe flow) can roughly be divided into two flows:
The transition between the two is when the Reynolds number is approximately **2,100**.

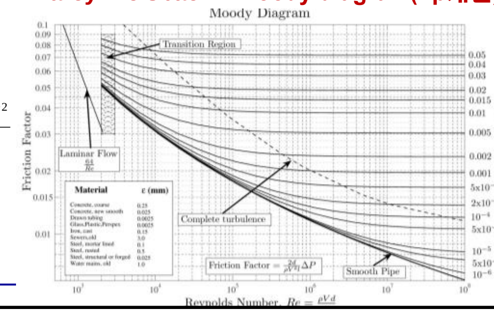
✓Laminar flow → Hagen-Poiseuille flow

$$Q = \frac{\pi R^4}{8 \mu l}(p_1 - p_2) = \frac{\pi d^4}{128 \mu l}(p_1 - p_2)$$

✓Turbulent flow→ Darcy-Weisbach→Moody diagram(Δp계산)

$$h = \lambda \frac{l}{d}\frac{V^2}{2g}$$

$$\Delta p = \gamma h = \lambda \frac{l}{d}\frac{\rho V^2}{2}$$



9.1 General External Flow Characteristics(외부유동 특성)

9.1 General External Flow Characteristics

✓외부유동(External flow)이란?

물체가 완전히 유체에 의해 둘러싸져 있는 유동(한쪽 경계가 없는 유동)

예: 단일 유체: 공기(자동차, 비행기 등)→공기역학(Aerodynamic)
이유체(Two fluids): 공기 및 물 (배, 해양 플랜트 구조물) →Froude number
일부분만 유체에 잠긴 경우: 빌딩, 해저 구조물 등

$Fr = \frac{\text{inertial force}}{\text{gravitational force}} = \frac{V}{\sqrt{gL}}$

✓외부유동(External flow)의 특징 및 주요 변수

경계층(Boundary layer) 형성: 물체 주변에 속도구배를 형성하는 층 (전단력 발생)

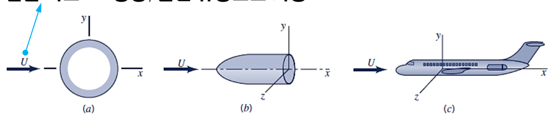
Drag and Lifting forces 형성: 속도구배에 의한 벽면 전단력(τ_w)에 기인(τ_w의 적분값)

- 항력(Drag force) : 벽면 전단응력(τ_w)의 적분값 (표면(마찰)항력+형상항력)
- 양력(Lifting force): 물체 형상의 비대칭성에 의해 발생하는 양력(압력분포 불균일)

9.1 General External Flow Characteristics

✓외부유동(External flow)의 분류 및 좌표축 설정

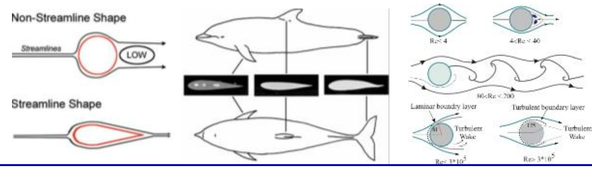
접근 속도 U : 정상, 균일 유동으로 가정



(a) two-dimensional (b) axisymmetric (c) three-dimensional

물체 형상에 따른 분류

- 유선형 물체(streamlined body): 물체 형상과 예상되는 유선이 일치하는 물체
- 비유선형 물체(blunt body): 물체 형상과 유선이 일치하지 않는 물체

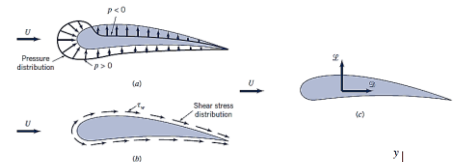


9.1.1 Lift and Drag Concepts

✓항력(drag force)과 양력(lifting force)의 발생

물체주변(경계층) 유동 특성:

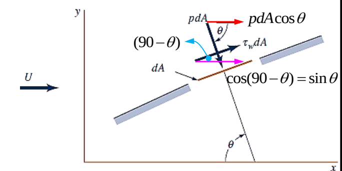
1) 물체 주변 속도 경계층 형성: 물체 표면에 벽면전단응력(τ_w) 발생 → 마찰항력 발생
2) 물체 형상에 따른 전후 압력차 발생 → 형상 항력 발생
3) 물체 형상에 따른 물체 주변 속도변화 → 압력변화에 의한 양력 발생



$\sin(90-\theta)=\cos\theta$

$\cos(90-\theta)=\sin\theta$

$$dF_x=(pdA)\cos\theta+(\tau_w dA)\sin\theta$$
$$dF_y=-(pdA)\sin\theta+(\tau_w dA)\cos\theta$$
$$D=\int dF_x=\int p\cos\theta dA+\int\tau_w\sin\theta dA$$
$$L=\int dF_y=-\int p\sin\theta dA+\int\tau_w\cos\theta dA$$

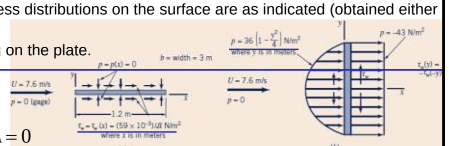


Example

Example 9.1 Drag from Pressure and Shear Stress Distributions

GIVEN: Air at standard conditions flows past a flat plate as is indicated in Fig. E9.1. In case (a) the plate is parallel to the upstream flow, and in case (b) it is perpendicular to the upstream flow. The pressure and shear stress distributions on the surface are as indicated (obtained either by experiment or theory).

FIND: Determine the lift and drag on the plate.



SOLUTION

(a) With the plate parallel to the upstream flow

$$L=-\int_{\text{top}}pdA+\int_{\text{bottom}}pdA=0$$

$$D=\int_{\text{top}}\tau_w dA+\int_{\text{bottom}}\tau_w dA=2\int_{\text{top}}\tau_w dA=2\int_{x=0}^{1.2m}\left(\frac{59\times10^{-3}}{x^{1/2}}N/m^2\right)(3m)dx=0.77N$$

(마찰항력)

(b) the plate perpendicular to the upstream flow

$$L=\int_{\text{front}}\tau_w dA-\int_{\text{back}}\tau_w dA=0$$

어떻게 구한 것이지?
(경계층 이론으로부터.....)

$$D=\int_{\text{front}}pdA-\int_{\text{back}}pdA=2\int_{y=-0.6}^{0.6}\left[36\left(1-\frac{y^2}{4}\right)N/m^2-(-43)N/m^2\right](3m)dy$$

$$=280.5N$$ (압력항력)

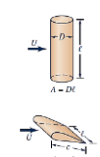
9.1.1 Lift and Drag Concepts

✓항력계수(drag coefficient, C_D) 와 양력계수(lifting coefficient, C_L)

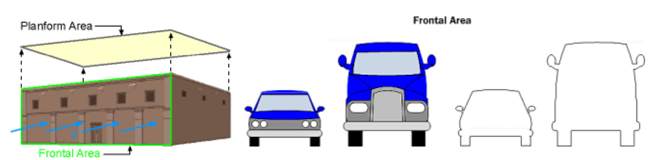
항력 및 양력계산에서 어려운 점은 압력 분포와 전단응력 분포를 얻는 것

따라서 보다 쉬운 항력 및 양력계수를 정의하여 사용

$$D=\int dF_x=\int p\cos\theta dA+\int\tau_w\sin\theta dA$$
$$L=\int dF_y=-\int p\sin\theta dA+\int\tau_w\cos\theta dA$$
$$C_L=\frac{L}{\frac{1}{2}\rho U^2 A},\quad C_D=\frac{D}{\frac{1}{2}\rho U^2 A}$$



여기서 A는 frontal area(투영면적, 정면도 면적)임에 유의



9.1.1 Lift and Drag Concepts

Ex: Boing 747-200의 양력 계산 예

보잉 747-200 이 고도 40,000ft (약12km)에서 순항(cruising) 할 때의 데이터

- 양쪽 날개 면적(S) (면적) : 511 m^2
- ρ(공기밀도) : 0.30267 kg/m^3 @40,000ft (12,192m)
- v (속도) : 순항속도 511 km/h (265.5 m/s)
- C_L (양력계수): 0.52 @2.4° 받음각
- Weight (중량) : 289 ton

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 A} \rightarrow L = C_L \times \frac{1}{2}\rho U^2 A$$

Lift = 0.52 x 511 m^2 x 0.5 x 0.30267 kg/m^3 x (265.5 m/s)^2
= 2,834,600 N (kg·m/sec^2)
= 289,245 kg$_f$
= 289 ton

9.1.2 Flow past an Object(물체주위 유동)

✓외부유동(External flow)에 영향을 미치는 변수

$$\mathrm{Re} = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$$

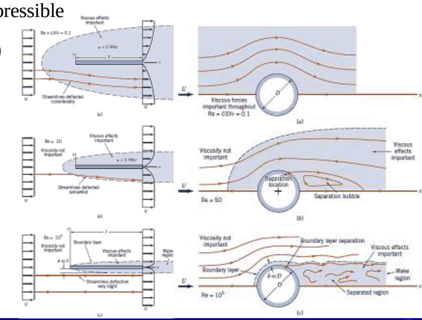
$$\mathrm{Re} = \frac{\rho VD}{\mu} = \begin{cases} \mathrm{Re} << 1: \text{Stokes flow (점성에 지배적 영향을 받음)} \\ \mathrm{Re} >> 1: \text{(관성에 지배적 영향을 받음)} \end{cases}$$

$$Ma = \frac{U}{c} = \begin{cases} Ma < 0.3: \text{incompressible } (0.01\text{m/s} < V < 100 m/s) \\ Ma > 0.3: \text{compressible} \end{cases}$$

$$Fr = \frac{U}{\sqrt{g\ell}} \text{ (for free surface)}$$

Re No. 증가(고속 유동)

1) 경계층이 얇아진다
(점성영역이 얇아진다)
2) 경계층 박리 발생
(유동박리 발생)
3) 후류 영역 확장
(유동 대칭성 사라짐)



9.2 Boundary Layer Characteristics (경계층의 특성)

Ludwig Prandtl
(1875– 1953), A German scientist.

In 1901 Prandtl became a professor of fluid mechanics at the technical school in Hannover, now the Technical University Hannover.
In 1904 he delivered a groundbreaking paper, **Fluid Flow in Very Little Friction**, in which he described the boundary layer and its importance for drag and streamlining. The paper also described flow separation as a result of the boundary layer, clearly explaining the concept of stall for the first time. The Prandtl number was named after him.

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{c_p \mu}{k}$$

ν : kinematic viscosity (m^2/s)
α : thermal diffusity $(m^2/s)(= k/\rho c_p)$
μ : dynamic visicosity $(Pa \cdot s)$
k : thermal conductivity $(W/(m \cdot K))$
c_p : specific heat $(J/(kg \cdot K))$
ρ : density (kg/m^3)

9.2 Boundary Layer Characteristics

✓Why is the boundary layer theory important in the fluid mechanics ?

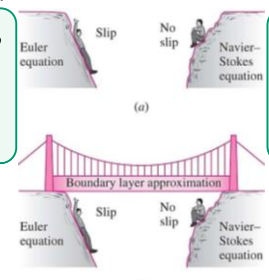
Euler Equation

$$\frac{D\vec{V}}{Dt} = \vec{f} - \frac{1}{\rho}\nabla p$$

$$\tau = \mu \frac{du}{dy} = 0$$

$$\nabla \times \vec{V} = 0$$

예제 9.1을 상기해보자
(D'alembert paradox)



N-S Equation

$$\frac{D\vec{V}}{Dt} = \vec{f} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{V} + \frac{1}{3}\nu\nabla(\nabla \cdot \vec{V})$$

$$= \vec{f} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{V} \text{ (for incomp.)}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}$$

Slip condition
No viscous effect
No rotation (curl$V$=0)
(potential flow)

simplification

No slip condition
With viscous effect
With rotation (cur$V$≠0)
Very weak rotation
Very high Re. No.

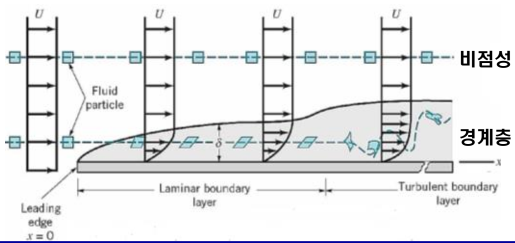
9.2 Boundary Layer Characteristics

✓**Prandtl's experimental results**

Reynolds No 증가 → 경계층이 얇아짐(대부분의 유동장이 관성력에 영향을 받음)
High Re No(관성력이 지배적)에서 고체 벽면 근처에서는 점성에 관계 없이 관성효과에 버금가는 점성효과가 크게 작용하는 얇은 경계층이 존재함을 입증

점성유동영역(Viscous flow)을 두 영역으로 나누어 해석할 수 있음을 설명 $\mathrm{Re}=\frac{\rho VL}{\mu}$

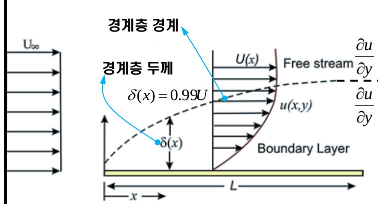
i) 경계층 영역 : 고체 경계면에 인접하여 점성효과가 현저한 영역
ii) 비점성 영역 : 점성효과를 무시할 수 있는 외곽 영역 (outer region, free stream) 즉, 유동을 비점성 유동으로 해석 (점성효과가 무시되는 영역)



9.2 Boundary Layer Characteristics

✓**경계층의 정의(Definition of Boundary Layer)**

고체 벽면 근처에서 점성에 관계 없이 관성효과에 버금가는 점성효과가 크게 나타나는 얇은 층(속도 경계층: 경계층 내 속도 분포는? 예제 9.1을 상기해보자)



No viscosity,
No rotation

$\frac{\partial u}{\partial y}=0$, Euler eq.(potential flow)

$\frac{\partial u}{\partial y}\neq 0$, N−S eq.(real flow)

With viscosity,
With rotation

Flow field
: Continuity eq.
: Euler eq.
: potential theory
Pressure
: Bernoulli eq.

Flow field
: Boundary Layer eq
: (N-S eq. approx.)
Pressure
: same within BL

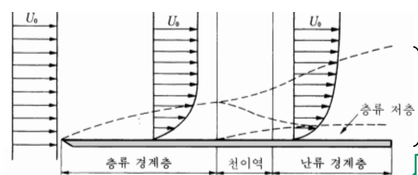
※최근에는 전체 유동장에 대해 N-S eq.을 직접 푼다

- **Introduce the concept of Boundary Layer**

이론(비점성유동)과 실제(점성유동)를 연결시켜 주는 매우 중요한 개념
N-S eq. 적용이 어려운 점성유동문제의 해석을 가능하게 해주는 실용적인 개념
경계층 개념의 도입은 현대유체역학의 시초로 인정됨

9.2 Boundary Layer Characteristics

✓**Growth of Boundary Layer (Boundary layer is a function of x-dir.)**



BL:
- pressure gradient
- surface roughness
- heat transfer
- turbulent

Laminar BL → Transition region → Turbulent BL → Logarithmic zone / Buffer zone / Laminar sub-layer : Law of wall region

$\mathrm{Re}_x=1\times10^5$ (Transition region)

$\mathrm{Re}_x\approx 2\times10^5 \sim 3\times10^6$ (Turbulent BL)

$0<y^+<5:\ u^+=y^+$

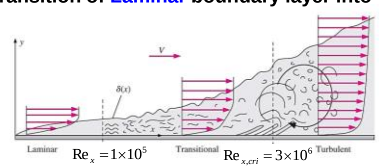
In BL, both inertia and viscosity forces are important equally→ Re No. is key variable

$$\mathrm{Re}_x=\frac{\rho Vx}{\mu}=\frac{Vx}{\nu}=\text{local Reynolds Number}$$

Presence of transition region: ex) when U=100m/s, BL is appear at **0.43 < $x(m)$ < 0.58**

9.2 Boundary Layer Characteristics

✓**Transition of Laminar boundary layer into Turbulent boundary layer (δ)**

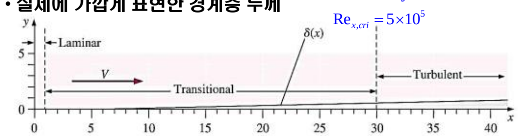


$\delta=\delta(x,V,\rho,\mu)$
$\mathbf{V}\uparrow \rightarrow \boldsymbol{\delta}\downarrow$
$\mathbf{Re}_x\uparrow \rightarrow \boldsymbol{\delta}\downarrow$

Re. No. on a flat plate $=\frac{\rho Vx}{\mu}=\frac{Vx}{\nu}$

- **실제에 가깝게 표현한 경계층 두께**

Critical Reynolds No.
$\mathrm{Re}_{x,cri}=5\times10^5$



Note : order of magnitude
$\begin{cases}\text{Chord of airfoil}: 1m \\ \delta(x)\text{ of BL}: 10^{-2}m \ll 1m\end{cases}$

Thickness of BL is too thin and the length of transient region is too long compared to laminar region.

경계층이 매우 얇다는 것은 Prandtl의 주요한 실험결과임 (매우 얇을 경우 물리적 현상은?)

9.2 Boundary Layer Characteristics

✓Thickness of boundary layer in the case of laminar & turbulent flow

$\delta \sim \sqrt{\nu t}$ (δ : thickness of BL or the length of momentum diffusivity)

$x = u_\infty t \;\rightarrow\; t = \dfrac{x}{u_\infty}$

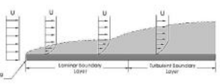
$\delta \sim \sqrt{\dfrac{\nu x}{u_\infty}} = \sqrt{\dfrac{\nu x^2}{u_\infty x}} (\leftarrow \mathrm{Re}_x = \dfrac{u_\infty x}{\nu})$ (뒷부분에서 설명됨)

$\dfrac{\delta}{x} \sim \dfrac{1}{\mathrm{Re}^{\frac{1}{2}}}$ or $\delta(x) \sim x\,\mathrm{Re}^{-\frac{1}{2}}$ (in the laminar flow)

$\dfrac{\delta}{x} \sim \dfrac{1}{\mathrm{Re}^{\frac{1}{5}}}$ or $\delta(x) \sim x\,\mathrm{Re}^{-\frac{1}{5}}$ (in the turbulent flow)

$\delta(x) = C_{\mathrm{laminar}}\, x\,\mathrm{Re}^{-\frac{1}{2}} = C_{\mathrm{lam}} \dfrac{x}{\mathrm{Re}^{1/2}}$

$\delta(x) = C_{\mathrm{turbulent}}\, x\,\mathrm{Re}^{-\frac{1}{5}} = C_{\mathrm{tur}} \dfrac{x}{\mathrm{Re}^{1/5}}$



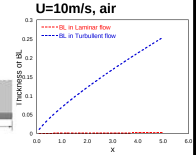
Air properties at 20ºC

$\rho_{air} = 1.205 (kg^* / m^3)$

$\mu_{air} = 2.0449 \times 10^{-5} (kg^* / m \cdot s)$

$\nu_{air} = 1.697 \times 10^{-5} (m^2 / s)$

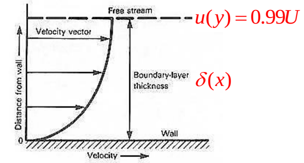
U=10m/s, air



9.2.1 BL Structure & Thickness on a Flat Plate

✓경계층 두께의 정의(Definition of Thickness of Boundary Layer)

고체 벽면($u_{y=0}(y)=0$) 으로부터 유체 점성의 영향이 미치는 곳($u(y)=0.99U$) 까지의 거리.
즉, 속도구배가 존재하는 영역의 두께: ($\delta(y)=0.99U$)



WHAT IS A FLUX?
The transfer of a quantity per unit area per unit time is called a flux.
The boundary layer plays a key role in transporting quantities such as:
mass flux , heat flux , moisture flux , momentum flux , and pollution fluxes

✓개념적 경계층 두께(Conceptual Thickness of Boundary Layer)

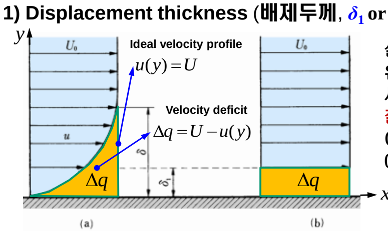
1) Displacement thickness (배제두께, δ_1 or δ^*)
2) Momentum thickness (운동량 두께, δ_2 or θ)
3) Dissipation energy thickness (소산에너지 두께, δ_3)

주의: 강의 노트에서는 부호를 혼용해서 사용

실제 경계층 두께(δ)는 점성 확산 효과를 다루는데 매우 유용
그러나 경계층 문제를 질량유량이나 운동량 플럭스 및 에너지와 관련시켜 다루기에 불편
따라서 개념적 두께인 배제두께(질량유량), 운동량 두께(운동량), 소산에너지 두께를 사용

9.2.1 BL Structure & Thickness on a Flat Plate

1) Displacement thickness (배제두께, δ_1 or δ^*)



실제 점성유동에서 경계층에 의해 배제된 유량에 상응하는 자유유동(free stream)에서 두께.
질량유량을 다루는 문제에서 배제두께를 이용, 이상유동 문제로 쉽게 다룰 수 있음
예) 풍동, 제트기관 유입부 설계

$(a)\;\; \Delta q = \int_0^\infty U_0 dy - \int_0^\infty u dy = \int_0^\infty (U_0 - u) dy$ (for unit width b)

$(b)\;\; \Delta q = \int_0^\infty U_0 dy - \int_{\delta_1}^\infty U_0 dy = \int_0^{\delta_1} U_0 dy = U_0 [y]_0^{\delta_1} = U_0 \delta_1$

$U_0 \delta_1 = \int_0^\infty (U_0 - u) dy$

$\delta_1 = \int_0^\infty \left(1 - \dfrac{u}{U_0}\right) dy \approx \int_0^\delta \left(1 - \dfrac{u}{U_0}\right) dy$

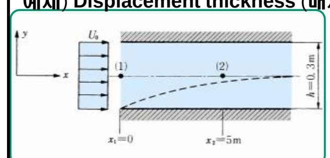
일반적으로 $\delta_1 = \dfrac{1}{3}\delta$

Simplying assumption

$\begin{cases} u = U_o & \text{at } y = \delta \\ \dfrac{\partial u}{\partial y} = 0 & \text{at } y = \delta \\ u << U_o & \text{at } y < \delta \end{cases}$

9.2.1 BL Structure & Thickness on a Flat Plate

예제) Displacement thickness (배제두께, δ_1 or δ^*)



Given:

$U_0 = 25 m/s$ (Inlet & centerline velocity)

$\dfrac{u}{U_0} = \left(\dfrac{y}{\delta}\right)^{1/7}$ (BL velocity distribution)

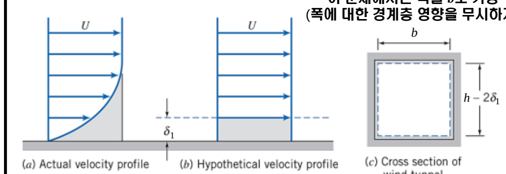
$\dfrac{\delta}{x} = 0.370 \left(\dfrac{\nu}{U_0 x}\right)^{1/5}$ (Turbulent BL thickness) (앞으로 구해야 할 공식)

$\nu = 1.45 \times 10^{-5} m^2/s$

Find: $(p_1 - p_2)_{x=5m} = ?$

Solution:

이 문제에서는 폭을 b로 가정
(폭에 대한 경계층 영향을 무시하자)



Assumptions
1) Steady flow
2) Incompressible flow
3) Uniform flow
4) Nofriction effect

9.2.1 BL Structure & Thickness on a Flat Plate

Step 1: Checking the $\delta(x)$

$$\delta = 0.370x\left(\frac{\nu}{U_0 x}\right)^{1/5} = 0.370\times 5(m)\left(\frac{1.45\times 10^{-5} m^2/s}{25m/s\times 5m}\right)^{1/5} = 0.0759m$$

$$\delta \langle \frac{h}{2} \ \text{(flow is under development)}$$

Step 2: basic equation (continuity eq.)

입구속도
입구단면적
경계층 제외 단면속도
경계층을 제외한 단면적 (배제두께 적용 시)

$$0 = \cancel{\frac{\partial}{\partial t}\int_{CV}\rho dV\!\!\!\!-} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = \int_{CS}\rho\vec{V}\cdot d\vec{A} = \rho U_0 A_0 = \rho U_2 A_2$$

From Bernoulli equation (경계층 밖은 속도 구배가 존재하지 않으므로 마찰력이 없음)

(배제두께를 이용하는 편리함)

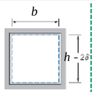
$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+\cancel{gz_1}=\frac{p_2}{\rho}+\frac{V_2^2}{2}+\cancel{gz_2}$$

$$p_1 - p_2 = \frac{\rho}{2}\left(V_2^2 - V_1^2\right) = \frac{\rho}{2}V_1^2\left(\left(\frac{V_2}{V_1}\right)^2 - 1\right)$$

$$= \frac{\rho}{2}\left(U_2^2 - U_0^2\right) = \frac{\rho}{2}U_0^2\left(\left(\frac{U_2}{U_0}\right)^2 - 1\right) = \frac{\rho}{2}U_0^2\left(\left(\frac{A_0}{A_2}\right)^2 - 1\right)$$

9.2.1 BL Structure & Thickness on a Flat Plate

From Continuity equation



$$\rho U_0 A_0 = \rho U_2 A_2$$

$$A_0 = bh,\ A_2 = b(h-2\delta_1)$$

$$\rho U_0 bh = \rho U_2 b(h-2\delta_1)$$

$$\frac{U_2}{U_0} = \frac{\rho bh}{\rho b(h-2\delta_1)} = \frac{h}{(h-2\delta_1)}$$

$$\eta = \frac{y}{\delta} \rightarrow dy = \delta d\eta$$ (매개변수 도입)

$$\delta_1 = \int_0^\delta\left(1-\frac{u}{U_0}\right)dy,\ \frac{u}{U_0} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$= \int_0^1\left(1-\eta^{1/7}\right)\delta d\eta$$

$$= \delta\left[\eta - \frac{\eta^{\frac{1}{7}+1}}{\frac{1}{7}+1}\right]_0^1$$

$$\delta_1 = \delta\left(1-\frac{7}{8}\right) = \frac{\delta}{8}$$

$$= \frac{0.0759}{8}m = 0.00949m$$

$$\frac{U_2}{U_0} = \frac{h}{h-2\delta_1} = \frac{0.3m}{0.3m - 2(0.00949m)}$$

$$= 1.0675m$$

$$p_1 - p_2 = \frac{\rho}{2}U_0^2\left(\left(\frac{A_0^2}{A_2^2}\right) - 1\right)$$

$$= \frac{1.23kg^*/m^3}{2}\times(25m/s)^2\times$$

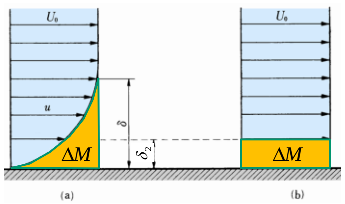
$$\left((1.0675)^2 - 1\right)\frac{N\cdot s^2}{kg\cdot m}$$

$$= 53.64Pa$$

9.2.1 BL Structure & Thickness on a Flat Plate

2) Momentum thickness (운동량 두께, δ_2 or θ)

경계층 생성으로 배제되는 운동량에 대응하는 비점성유동장의 두께



실제유동질량

$$\dot{m} = \int_0^\infty \rho ubdy$$

실제유동질량의 실제 운동량

$$M_{real} = u\int_0^\infty \rho ubdy$$

실제유동질량의 자유유동 운동량

$$M_{free} = U_0\int_0^\infty \rho ubdy$$

$$(a)\ \text{momentum loss}\ (\Delta M) = U_0\int_0^\infty \rho ubdy - u\int_0^\infty \rho ubdy = \int_0^\infty \rho u(U_0 - u)bdy$$

$$(b)\ \text{momentum}\ (\Delta M) \quad = \rho U_0^2 b\delta_2$$

$$\rho U_0^2 b\delta_2 = \int_0^\infty \rho u(U_0 - u)bdy$$

$$\delta_2 = \int_0^\infty \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy \approx \int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy$$

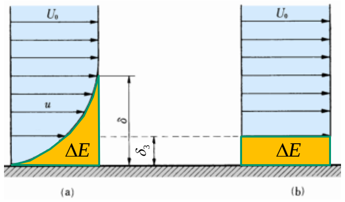
Note: 배제두께

$$\delta_1 = \int_0^\delta\left(1-\frac{u}{U_0}\right)dy$$

9.2.1 BL Structure & Thickness on a Flat Plate

3) Dissipation energy thickness (소산에너지 두께, δ_3)

경계층 내 유동 감속으로 인하여 배제되는 에너지손실 양에 상응하는 자유유동 두께



(a)에서 단위질량당 에너지 손실

$$h_L = \left(\frac{p_0}{\rho}+\frac{U_0^2}{2}\right) - \left(\frac{p}{\rho}+\frac{u^2}{2}\right)$$

폭 dy를 통과하는 유동질량에 대한 에너지 손실

$$\rho ubdyh_L = \left\{\left(\cancel{\frac{p_0}{\rho} - \frac{p}{\rho}}\right) + \left(\frac{U_0^2}{2} - \frac{u^2}{2}\right)\right\}\rho ubdy$$

$$= \frac{\rho}{2}\left(U_0^2 - u^2\right)ubdy$$ (압력차가 매우 작으므로)

$$(a)\ \text{energy loss}(\Delta E) = \int_0^\infty \rho ubh_L dy = \int_0^\infty \frac{1}{2}\left(U_0^2 - u^2\right)\rho ubdy = \frac{1}{2}\rho\int_0^\delta\left(U_0^2 - u^2\right)ubdy$$

$$(b)\ \text{energy}(\Delta E) \quad = \frac{1}{2}\rho U_0^3 b\delta_3\left(\leftarrow \frac{1}{2}mV^2 = \frac{1}{2}\rho AVV^2 = \frac{1}{2}\rho V^3 b\delta_3\right)$$

$$\frac{1}{2}\rho U_0^3 b\delta_3 = \frac{1}{2}\rho\int_0^\delta\left(U_0^2 - u^2\right)ubdy \rightarrow \delta_3 = \int_0^\delta \frac{u}{U_0}\left(1-\left(\frac{u}{U_0}\right)^2\right)dy$$

Note: 두께 비교

$$\delta_1 = \int_0^\delta\left(1-\frac{u}{U_0}\right)dy$$

$$\delta_2 = \int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy$$

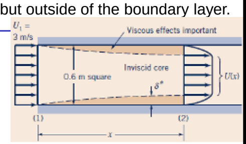
Example

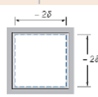
Example 9.3 Boundary Layer Displacement Thickness

GIVEN: Air flowing into a 0.6 m square duct with a uniform velocity of 3 m/s forms a boundary layer on the walls. The fluid within the core region 1outside the boundary layers 2 flows as if it were inviscid. From advanced calculations it is determined that for this flow the boundary layer displacement thickness is given by $\delta^* = 0.004(x)^{1/2}$ where $\delta^*$and x are in meters

FIND: Determine the velocity $U=U(x)$ of the air within the duct but outside of the boundary layer.

SOLUTION





$$U_1 A_1 = 3m/s(0.6m)^2 = 1.1m^3/s = \int_{(2)} udA$$

$$1.1m^3/s = \int_{(2)} udA = U(0.6m - 2\delta^*)^2$$

$$1.1m^3/s = U(0.6m - 0.008x^{1/2})^2$$

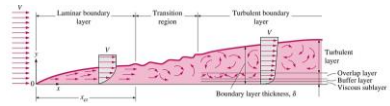
$$U = \frac{1.1}{(0.6m - 0.008x^{1/2})^2} m/s$$

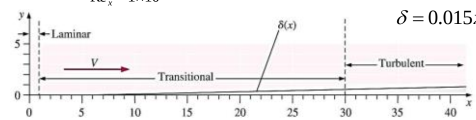
$$p_1 + \frac{1}{2}\rho U_1^2 = p_2 + \frac{1}{2}\rho U_2^2,\ (p_1 = 0, \rho = 1.2kg/m^3)$$

$$p_2 = \frac{1}{2}\rho\left(U_1^2 - U_2^2\right) = \frac{1}{2}\rho\left((3m/s)^2 - \frac{1.2}{(0.6m - 0.008x^{1/2})^4} m/s\right)$$

$$= -2.12 N/m^2 (\text{at } x = 30cm)$$

Motion Equation in Boundary layer (경계층 운동방정식)





Note ;
rough-estimated magnitude (L vs δ)

for laminar flow ($Re_x < 10^5$)

$$\frac{\delta}{x} = \frac{5}{Re^{0.5}} \quad \text{for } Re = 10^5$$

$\delta = 0.015x$ **(Namely, about 1.5% of x)**

Basic assumption for BL approximation is that the BL is very thin.

9.2 Motion Equation in Boundary layer

✓N-S Equation for incompressible flow

Continuity eq. : $\nabla\cdot\vec{V} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$ (for incompressible flow)

Momentum eq. : $\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V} + \frac{1}{3}\mu\nabla\underbrace{(\nabla\cdot\vec{V})}_{=0}$

1) N-S eq. in 2D flow and Basic assumptions

i) 2D flow, ii) steady flow, iii) No body force iv) incompressible flow

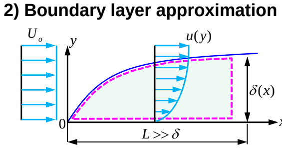
Continuity eq. : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \cdots(1)$$

Momentum eq. : $$x-\text{dir}:\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \cdots(2)$$

$$y-\text{dir}:\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \cdots(3)$$

9.2 Motion Equation in Boundary layer

2) Boundary layer approximation (경계층 근사: 중요)



$$\left.\begin{array}{l}\frac{\partial^2}{\partial x^2} \sim \frac{1}{L^2} \\ \frac{\partial^2}{\partial y^2} \sim \frac{1}{\delta^2}\end{array}\right\} \rightarrow \frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$$

Momentum eq. : ② (x-dir) $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

Continuity eq. : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{u}{L} + \frac{v}{\delta} \sim 0 \rightarrow v \approx -\frac{\delta}{L}u$$

$$\rightarrow v \sim \overset{\approx 0}{\frac{\delta}{L}}u \rightarrow v \ll u$$

Momentum eq. : ③ (y-dir) $$\cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

즉, $p=p(x)$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} \sim 0 \rightarrow \therefore \frac{\partial p}{\partial y} \sim 0 \rightarrow \frac{\partial p}{\partial x} = \frac{dp}{dx}$$

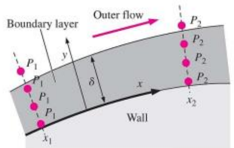
9.2 Motion Equation in Boundary layer

3) Pressure within BL

$$\frac{\partial p}{\partial x}=\frac{dp}{dx} \rightarrow p=p(x)$$

Summary of BL approximation:

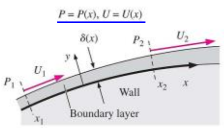
$$\frac{\partial^2}{\partial x^2} \langle\langle \frac{\partial^2}{\partial y^2},\ v\langle\langle u,\ \frac{\partial p}{\partial y}\sim 0 \rightarrow p=p(x)$$



Pressure change across a BL is negligible

→ Applicable to asses the pressure of outer flow

$$\frac{\partial p}{\partial y}\sim 0 \rightarrow \frac{\partial p}{\partial x}=\frac{dp}{dx}$$



Pressure across a BL is the same as out flow,

We may apply Bernoulli eq. to BL equation in x-dir.

$$\frac{p}{\gamma}+\frac{1}{2g}U^2+\cancel{z}=C$$

$$\frac{p}{\rho}+\frac{1}{2}U^2=C$$

$$\frac{1}{\rho}\frac{dp}{dx}+U\frac{dU}{dx}=0 \rightarrow \frac{1}{\rho}\frac{dp}{dx}=-U\frac{dU}{dx}$$

9.2 Motion Equation in Boundary layer

✓Motion Equation in Boundary layer (경계층운동방정식)

- Continuity eq. : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ $\quad\left(\frac{\partial^2}{\partial x^2} \langle\langle \frac{\partial^2}{\partial y^2},\ v\langle\langle u,\ \frac{\partial p}{\partial y}\sim 0\right)$
- Momentum eq. : $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\ \left(-\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx}\right)$

도입된 가정: 경계층은 매우 얇다 ($\delta << L$)

Note: 2D N-S Equation for Incompressible Flow

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \cdots(1)$$

- Continuity eq. : $x-\text{dir}:\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\cdots(2)$
- Momentum eq. : $y-\text{dir}:\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\cdots(3)$

9.2 Motion Equation in Boundary layer

✓Motion Equation in Boundary layer (경계층운동방정식)

- Continuity eq. : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ $\quad\left(\frac{\partial^2}{\partial x^2} \langle\langle \frac{\partial^2}{\partial y^2},\ v\langle\langle u,\ \frac{\partial p}{\partial y}\sim 0\right)$
- Momentum eq. : $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\ \left(-\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx}\right)$

Boundary Layer Equation (경계층 운동 방정식)

- Parabolic type(포물선 방정식)
- 유동정보는 하류에서 상류로 전파 (즉, 하류로 정보가 전파되지 않음. 초기값이 필요)
- 3면의 경계조건만 필요. 문제풀기가 매우 쉬어짐

Navier-Stokes Equation

- Elliptic type(타원형 방정식)
- 유동정보가 공간 전체에 전파 (경계값 문제가 됨)
- 4면의 유동정보가 필요. 문제풀기가 매우 복잡

(몇 가지 단순한 경우를 제외하고 대수 해를 구하기 어려움: 모델링 및 수치 해가 존재)

Note: Parabolic and elliptic type of PDE

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

$B^2-4AC=0$: Parabolic type

$B^2-4AC<0$: Elliptic type

9.2 Motion Equation in Boundary layer

✓Solution of Boundary Layer Eq.(경계층운동방정식의 해)

- Continuity eq. : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ $\quad$ Pressure gradient term
- Momentum eq. : $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\ \left(-\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx}\right)$

1) Exact solution for Laminar Flow : Prandtl & Blasius' solution

2) Approximate solution : Karman-Pohlhausen's Method

(called as Von Karman's Momentum Integral Method)

i) for Laminar and Turbulent

ii) for without pressure gradient($dp/dx=0$) and With Pressure gradient($dp/dx\neq 0$)

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq 0$	$dp/dx=0$	$dp/dx\neq 0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIM)	O	O	O	O

Exact Solution of BL Eq.
(Blasius Solution for Laminar BL)

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq 0$	$dp/dx=0$	$dp/dx\neq 0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIE)	O	O	O	O

Exact Solution Method: Blasius' solution

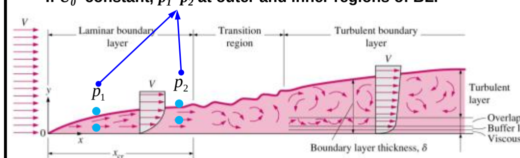
✓**Exact Solution**: in the case of pressure gradient $dp/dx=0$

Assumption : $U_0 = \text{Constant},\ \dfrac{dp}{dx} = 0$ (No pressure gradient)(see below schematics)

Boundary layer equations :

- **Continuity eq. :** $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$
- **Momentum eq. :** $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2}$ $\quad \text{BC}: \begin{cases} u(x,0) = v(x,0) = 0 & \text{at } y = 0 \\ u(x,\infty) = U_0 & \text{at } y = \infty \end{cases}$

If U_0=constant, p_1=p_2 at outer and inner regions of BL.



Note: Euler's eq.

$$\frac{p}{\rho} + \frac{1}{2}U^2 = C$$

$$\frac{1}{\rho}\frac{dp}{dx} + U\frac{dU}{dx} = 0$$

$$\frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx} = 0$$

Exact Solution Method: Blasius' solution

Blasius performed a transformation technique to change the set of two partial differential equations(**PDE**) into a single ordinary differential equation(**ODE**). He solved the boundary layer over a flat plate in external flows. (No pressure gradient)

To solve the PDE, introduce similarity parameter, η

$$\frac{\delta}{x} \propto \text{Re}_x^{-\frac{1}{2}} \text{ for laminar flow} \rightarrow \frac{\delta}{x} \propto \sqrt{\frac{\nu}{U_0 x}} \rightarrow \delta \sim \sqrt{\frac{\nu x}{U_0}}$$

(차원해석 결과)

Blasius defined new non-dimensional similarity parameter(η) as;

$$\eta \sim \frac{y}{\delta} \rightarrow \eta = y\sqrt{\frac{U_0}{\nu x}} \quad \left(\eta = \frac{y}{\delta} = y \times \frac{1}{\delta} = y\sqrt{\frac{U_0}{\nu x}}\right)$$

Introduce 2D stream function defined as ; $u = \dfrac{\partial \psi}{\partial y},\ v = -\dfrac{\partial \psi}{\partial x}$

Rewrite continuity eq. using the stream function,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\text{by chain rule}} = 0$$

This result shows that the stream function is satisfied with continuity equation automatically

Exact Solution Method: Blasius' solution

Dimension of ψ may be determined as the multiplication of velocity and distance

$$\left.\begin{aligned} &\text{Velocity} : U_0 \\ &\text{Distance} : \delta \sim \sqrt{\frac{\nu x}{U_0}} \end{aligned}\right\} \rightarrow \psi \sim U_0\sqrt{\frac{\nu x}{U_0}} \sim \sqrt{U_0 \nu x}$$

Note: $\boxed{u = \dfrac{\partial \psi}{\partial y} \rightarrow \partial \psi = u \times \partial y}$

$$\therefore\ \psi = \sqrt{U_0 \nu x}\, f(\eta) \rightarrow f(\eta) = \frac{\psi}{\sqrt{U_0 \nu x}} \text{ (Non-dimensional stream function)}$$

Summary

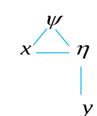
$$\psi = \sqrt{U_0 \nu x}\, f(\eta) \rightarrow \psi = f(x, \eta)$$

$\eta = y\sqrt{\dfrac{U_0}{\nu x}} \rightarrow \eta = f(x, y)$($\eta$ transform two parameters (x, y) into one parameter(η))

by chain rule

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta} = -v$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial \psi}{\partial \eta} = u$$



Note:

$$\psi = f(\eta)\sqrt{U_0 \nu x} \qquad \eta = y\sqrt{\frac{U_0}{\nu x}}$$

$$\frac{\partial \psi}{\partial \eta} = f'(\eta)\sqrt{U_0 \nu x} \qquad \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_0}{\nu x}}$$

Exact Solution Method: Blasius' solution

$$u=\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y}=\left(f'(\eta)\sqrt{U_0\nu x}\right)\sqrt{\frac{U_0}{\nu x}}=\underline{f'(\eta)}U_0$$

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial}{\partial x}\left(f'(\eta)U_0\right)=\frac{\partial U_0}{\partial x}f'(\eta)+U_0\frac{\partial}{\partial x}f'(\eta)\\&=0+U_0\frac{\partial f'(\eta)}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}=U_0 f''(\eta)\frac{\partial\eta}{\partial x}\ (\because\ U_0=\text{const})\\&=U_0 f''(\eta)\left(-\frac{\eta}{2x}\right)=\underline{-\frac{\eta}{2x}U_0 f''(\eta)}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y}&=\frac{\partial}{\partial y}\left(f'(\eta)U_0\right)\\&=\frac{\partial U_0}{\partial y}f'(\eta)+U_0\frac{\partial f'(\eta)}{\partial y}\\&=0+U_0\frac{\partial f'(\eta)}{\partial y}\cdot\frac{\partial\eta}{\partial y}=\underline{U_0 f''(\eta)\sqrt{\frac{U_0}{\nu x}}}\end{aligned}$$

$$\psi(x,\eta)=\sqrt{U_0\nu x}\,f(\eta)$$

$$\eta(x,y)=y\sqrt{\frac{U_0}{\nu x}}$$

Note :

$$\begin{aligned}\frac{\partial\eta}{\partial x}&=\frac{\partial}{\partial x}\left(y\sqrt{\frac{U_0}{\nu x}}\right)\\&=y\sqrt{\frac{U_0}{\nu}}\frac{\partial}{\partial x}\left(x^{-\frac{1}{2}}\right)\\&=y\sqrt{\frac{U_0}{\nu}}\left(-\frac{1}{2x}\frac{1}{\sqrt{x}}\right)\\&=-y\sqrt{\frac{U_0}{\nu x}}\frac{1}{2x}\\&=-\frac{\eta}{2x}\end{aligned}$$

$$\left(x^{-\frac{1}{2}}\right)'=-\frac{1}{2x\sqrt{x}}$$

Exact Solution Method: Blasius' solution

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}&=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(U_0 f''(\eta)\sqrt{\frac{U_0}{\nu x}}\right)\\&=\frac{\partial U_0}{\partial y}f''(\eta)\sqrt{\frac{U_0}{\nu x}}+U_0\sqrt{\frac{uU_0}{\nu x}}\frac{\partial}{\partial y}f''(\eta)\\&=0+U_0\sqrt{\frac{U_0}{\nu x}}\frac{\partial f''(\eta)}{\partial\eta}\frac{\partial\eta}{\partial y}=U_0\sqrt{\frac{U_0}{\nu x}}f'''(\eta)\sqrt{\frac{U_0}{\nu x}}\\&=\underline{\frac{U_0^{\ 2}}{\nu x}}f'''(\eta)\end{aligned}$$

Note: Stream function

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial x}+\frac{\partial\eta}{\partial x}\frac{\partial\psi}{\partial\eta}=-v$$

$$\frac{\partial\psi}{\partial y}=\frac{\partial\eta}{\partial y}\frac{\partial\psi}{\partial\eta}=u$$

$$\begin{aligned}v&=-\frac{\partial\psi}{\partial x}=-\left(\frac{\partial\psi}{\partial x}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\psi}{\partial\eta}\right)\\&=-\left\{\frac{\partial}{\partial x}\left(\sqrt{U_0\nu x}\,f(\eta)\right)+\left(-\frac{\eta}{2x}\right)\left(f'(\eta)\sqrt{U_0\nu x}\right)\right\}\end{aligned}$$

Exact Solution Method: Blasius' solution

$$\begin{aligned}&=-\sqrt{U_0\nu}\frac{1}{2\sqrt{x}}f(\eta)+\frac{\eta}{2x}f'(\eta)\sqrt{U_0\nu x}\\&=-\sqrt{U_0\nu x}\frac{1}{2x}f(\eta)+\frac{\eta}{2x}f'(\eta)\sqrt{U_0\nu x}\\&=\frac{\sqrt{U_0\nu x}}{2x}\left(\eta f'(\eta)-f(\eta)\right)\\&=\underline{\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\left(\eta f'(\eta)-f(\eta)\right)}\end{aligned}$$

Summary

$$u=f'(\eta)U_0\ \rightarrow\ f'(\eta)=\frac{u}{U_0},\qquad v=\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\{\eta f'(\eta)-f(\eta)\}$$

$$\frac{\partial u}{\partial x}=-\frac{\mu}{2x}U_0 f''(\eta),\quad \frac{\partial u}{\partial y}=U_0 f''(\eta)\sqrt{\frac{U_0}{\nu x}},\quad \frac{\partial^2 u}{\partial y^2}=\frac{U_0^{\ 2}}{\nu x}f'''(\eta)$$

Exact Solution Method: Blasius' solution

Substitute these results into momentum equation

$$\left.\begin{aligned}&u=f'(\eta)U_0\ \rightarrow\ f'(\eta)=\frac{u}{U_0},\qquad v=\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\{\eta f'(\eta)-f(\eta)\}\\&\frac{\partial u}{\partial x}=-\frac{\mu}{2x}U_0 f''(\eta),\quad \frac{\partial u}{\partial y}=U_0 f''(\eta)\sqrt{\frac{U_0}{\nu x}},\quad \frac{\partial^2 u}{\partial y^2}=\frac{U_0^{\ 2}}{\nu x}f'''(\eta)\end{aligned}\right\}\rightarrow\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

And then rearrangement the equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$f'(\eta)U_0\times\left(-\frac{\eta}{2x}U_0 f''(\eta)\right)+\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\{\eta f'(\eta)-f(\eta)\}\times U_0 f''(\eta)\sqrt{\frac{U_0}{\nu x}}=\nu\times\frac{U_0^{\ 2}}{\nu x}f'''(\eta)$$

$$-\frac{\eta U_0^{\ 2}}{2x}f'\,f''+\left(\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\eta f'-\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}f\right)U_0 f''\sqrt{\frac{U_0}{\nu x}}=\frac{U_0^{\ 2}}{x}f'''$$

$$-\frac{\eta U_0^{\ 2}}{2x}f'\,f''+\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\eta f'U_0 f''\sqrt{\frac{U_0}{\nu x}}-\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}fU_0 f''\sqrt{\frac{U_0}{\nu x}}=\frac{U_0^{\ 2}}{x}f'''$$

Exact Solution Method: Blasius' solution

$$\underbrace{-\frac{\eta U_0^2}{2x}f'f''+\frac{1}{2}\frac{U_0^2}{x}\eta f'f''}_{=0}-\frac{1}{2}\frac{U_0^2}{x}f'f''=\frac{U_0^2}{x}f'''$$

$$-\frac{1}{2}\frac{U_0^2}{x}f'f''=\frac{U_0^2}{x}f'''$$

$$-\frac{1}{2}f'f''=f'''$$

Therefore, the original x-momentum can be written as below upon simplifications. Note that this is an ordinary differential equation with η as the independent variable and $f(\eta)$ is the dependent variable.

$$\therefore \boxed{2f'''+f\,f''=0 \quad \text{or} \quad 2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=0}$$

3rd-order non-linear ordinary differential eq. (ODE) instead of PDE

Boundary conditions

$$\left.\begin{array}{l}\psi=0 \text{ at } y=0\\ \dfrac{\partial\psi}{\partial y}=0 \text{ as } y=0\end{array}\right\} f(0)=f'(0)=0$$

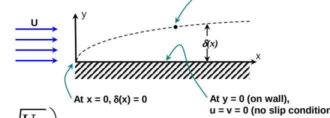
$$\frac{\partial\psi}{\partial y}=U \text{ at } x=0$$

$$\frac{\partial\psi}{\partial y}=U \text{ as } y=\infty$$

Exact Solution Method: Blasius' solution

To solve the 3rd-order non-linear ordinary differential eq. (ODE) we need three boundary conditions.

$$2f'''+f\,f''=0$$



Transform BCS on x, y into on η

$$BCS:\left\{\begin{array}{lll}(1)\ y=0 & \eta=0\left(\eta=\dfrac{y}{\delta}=y\sqrt{\dfrac{U_0}{\nu x}}\right) & \\ \quad u=v=0 & f'=0\left(u=0=U_0 f'\rightarrow f'=0\right)\rightarrow f'(0)=f(0)=0 & \\ \quad \psi=0 & f=0\left(\psi=f(\eta)\sqrt{\nu x U_0}=0 \quad \rightarrow f=0\right) & \\ (2)\ y=\infty & \eta=\infty\left(\eta=\dfrac{y}{\delta}=y\sqrt{\dfrac{U_0}{\nu x}}\right) & \\ \quad u=U_0 & f'=1\left(u=U_0=U_0 f' \ \therefore f'=1\right)\rightarrow f'(\infty)=1 & \end{array}\right.$$

by using 4th Runge-Kutta integral method, we may get the solution

Exact Solution Method: Blasius' solution

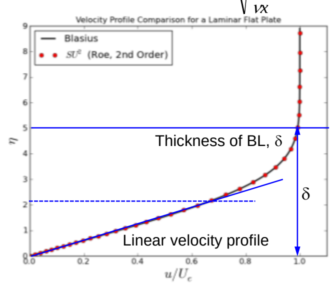
Blasius solution for a flat plate

η	f	f'	f''
0.0	0.00000	0.00000	0.33206
0.2	0.00664	0.06641	0.33198
0.4	0.02656	0.13276	0.33147
0.6	0.05973	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32978	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45626	0.30787
1.6	0.42032	0.51676	0.29666
1.8	0.52952	0.57476	0.28293
2.0	0.65002	0.62977	0.26675
2.2	0.78119	0.68131	0.24835
2.4	0.92229	0.72898	0.22809
2.6	1.07251	0.77245	0.20645
2.8	1.23098	0.81151	0.18401
3.0	1.39681	0.84604	0.16136
3.2	1.56909	0.87608	0.13913
3.4	1.74695	0.90176	0.11788
3.6	1.92952	0.92333	0.09809
3.8	2.11603	0.94112	0.08013
4.0	2.30575	0.95552	0.06423
4.2	2.49804	0.96696	0.05052
4.4	2.69236	0.97587	0.03897
4.6	2.88825	0.98268	0.02948
4.8	3.08532	0.98779	0.02187
5.0	3.28327	0.99154	0.01591
5.2	3.48187	0.99425	0.01134
5.4	3.68092	0.99616	0.00793
5.6	3.88029	0.99748	0.00543
5.8	4.07988	0.99838	0.00365
6.0	4.27962	0.99897	0.00240
6.2	4.47946	0.99936	0.00155
6.4	4.67936	0.99961	0.00098
6.6	4.87929	0.99977	0.00061
6.8	5.07926	0.99986	0.00037
7.0	5.27924	0.99992	0.00022
7.2	5.47923	0.99996	0.00013
7.4	5.67922	0.99998	0.00007
7.6	5.87922	0.99999	0.00004
7.8	6.07921	0.99999	0.00002
8.0	6.27921	1.00000	0.00001

$$\eta(x,y)=\frac{y}{\delta}=y\sqrt{\frac{U_0}{\nu x}},\quad f(\eta)=\frac{\psi}{\sqrt{U_0\nu x}},\quad f'(\eta)=\frac{u}{U_0}$$

When $\eta=5,\ \dfrac{df}{d\eta}=f'=\dfrac{u}{U_0}=0.99$

$$\eta_{=5}(x,y=\delta)=\frac{\delta}{\delta}=\delta\sqrt{\frac{U_0}{\nu x}}\ \rightarrow\ \delta=\frac{5}{\sqrt{\dfrac{U_0}{\nu x}}}$$



Exact Solution Method: Blasius' solution

Using the alternate definition of η, we may get the thickness of boundary layer:

$$\eta=\frac{y}{\delta}=y\sqrt{\frac{U_0}{\nu x}}\ \rightarrow\ \eta_{=5}(x,y=\delta)=\frac{\delta}{\delta}=\delta\sqrt{\frac{U_0}{\nu x}}\ \rightarrow\ \delta=\frac{\eta=5}{\sqrt{\dfrac{U_0}{\nu x}}}\ \rightarrow\ \frac{\delta}{x}=\frac{5}{\sqrt{\dfrac{U_0 x}{\nu}}}=\frac{5}{\mathrm{Re}_x^{\frac{1}{2}}}=\frac{5}{\sqrt{\mathrm{Re}_x}}$$

The wall shear stress, τ_w may be written as:

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}+\overset{(\because v<<u)}{\cancel{\frac{\partial v}{\partial x}}}\right)_{y=0}=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\mu U_0\sqrt{\frac{U_0}{\nu x}}\left.\frac{d^2f}{d\eta^2}\right]_{\eta=0}=\mu U_0\sqrt{\frac{U_0}{\nu x}}\times\overset{\text{from table}}{0.332}=0.332\frac{\rho U_0^2}{\sqrt{\mathrm{Re}_x}}$$

The skin friction coefficient is defined as the non-dim. wall shear stress:

$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U_0^2}=\frac{0.332\dfrac{\rho U_0^2}{\sqrt{\mathrm{Re}_x}}}{\frac{1}{2}\rho U_0^2}=\frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

The overall skin friction coefficient:

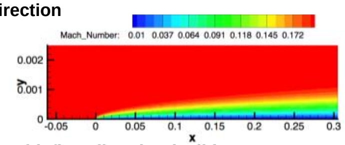
$$\bar{C}_f=\frac{1}{L}\int_0^L C_f\,dx=2C_f=\frac{1.328}{\sqrt{\mathrm{Re}_x}}$$

Exact solution	Approximate sol.
$\dfrac{\delta}{x}=\dfrac{5}{\sqrt{\mathrm{Re}_x}}$	$\dfrac{\delta}{x}=\dfrac{3.46}{\sqrt{\mathrm{Re}_x}}$
$C_f=\dfrac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_f=\dfrac{0.577}{\sqrt{\mathrm{Re}_x}}$
$\bar{C}_f=\dfrac{1.328}{\sqrt{\mathrm{Re}_x}}$	$\bar{C}_f=\dfrac{1.154}{\sqrt{\mathrm{Re}_x}}$

Exact Solution Method: Blasius' solution

Growth of the boundary layer with x-direction

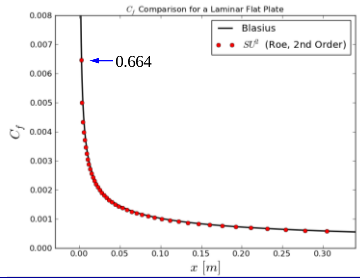
$$\frac{\delta}{x}=\frac{5}{\sqrt{\mathrm{Re}_x}}$$



Variation of the skin friction coefficient with flow direction (x-dir)

$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U_0^2}=\frac{0.664}{\sqrt{\mathrm{Re}_x}}=\frac{0.664}{\sqrt{\frac{U_o x}{\nu}}}$$

C_f decreases with the growth of the Boundary Layer



9.2.2 Prandtl/Blasius Exact Solution

✓N-S equation for 2D, Steady, Laminar flow without Body force(중력 무시)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

• 2D Continuity eq.: $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

타원형 방정식(elliptic equations)

✓Prandtl's idea for high Re. No. flow

경계층은 매우 얇기 때문에 평판에서 수직인 속도성분은 평행인 속도성분보다 매우 작다.
경계층에서 모든 매개변수의 수직방향으로 변화율은 유동방향의 변화율보다 훨씬 크다.
평판 위의 경계층 유동에서 압력은 유체전반에 걸쳐 일정하다.

$v<<u$ and $\frac{\partial}{\partial x}<<\frac{\partial}{\partial y}$ (Prandtl이 제안한 아이디어), $\frac{\partial p}{\partial x}=\frac{\partial p}{\partial y}=0$

✓경계층 방정식 (단순화된 N-S eq. for 2D, Steady, Laminar flow, without Body force)

$$\left.\begin{aligned}&\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0\\&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}\end{aligned}\right\}\mathrm{BCs}:\begin{cases}u=v=0 & \text{at } y=0\\ u=U & \text{at } y=\infty\end{cases}$$

포물선 방정식(parabolic equations)

9.2.2 Prandtl/Blasius Exact Solution

✓Blasius solution for boundary equation(경계층 방정식), Prandtl의 제자 (1908)

무차원으로 나타낸 평판의 경계층 속도분포: 위치에 관계없이 동일.
차원해석을 통해 관계식을 구할 수 있음.

$$\frac{u}{U}=g\left(\frac{y}{\delta}\right)\rightarrow\delta\sim\left(\frac{\nu x}{U}\right)^{1/2}$$

경계층 내에서는 점성력과 관성력이 평형
속도구배는 유동방향보다 수직방향이 더 큼

Blasius는 경계층 방정식(비선형 편미방)을 풀기 위해 무차원 상사변수와 유동함수를 도입

$$\eta=\left(\frac{U}{\nu x}\right)^{1/2}y,\ \psi=(\nu,\,x,\,U)^{1/2}f(\eta)\leftarrow u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x}$$

$$u=Uf'(\eta)\rightarrow v=\left(\frac{\nu U}{4x}\right)^{1/2}(\eta f'-f)$$

위 두 식을 경계층 방정식에 대입하여 정리하면 비선형 3차 상미분방정식을 얻을 수 있음

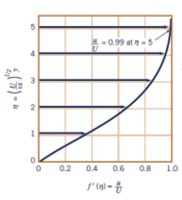
$$2f'''+ff''=0,\ BCs:\begin{cases}f=f'=0 & \text{at } \eta=0\\ f'\rightarrow 1 & \text{as } \eta\rightarrow\infty\end{cases}$$

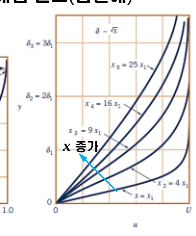
9.2.2 Prandtl/Blasius Exact Solution

상사변수로 나타낸 비선형 3차 상미분방정식의 해: 수치해법 필요(엄밀해)

Laminar Flow along a Flat Plate (the Blasius Solution)

$\eta=y(U/\nu x)^{1/2}$	$f'(\eta)=u/U$	η	$f'(\eta)$
0	0	3.6	0.9233
0.4	0.1328	4.0	0.9555
0.8	0.2647	4.4	0.9759
1.2	0.3938	4.8	0.9878
1.6	0.5168	5.0	0.9916
2.0	0.6298	5.2	0.9943
2.4	0.7290	5.6	0.9975
2.8	0.8115	6.0	0.9990
3.2	0.8761	∞	1.0000





$$f'(\eta)=\frac{u}{U}=0.9916\ \text{at}\ \eta=5.0\ \rightarrow\ \delta=5\sqrt{\frac{\nu x}{U}}\ \text{or}\ \frac{\delta}{x}=\frac{5}{\sqrt{\mathrm{Re}_x}}$$

$$\frac{\delta^*}{x}=\frac{1.721}{\sqrt{\mathrm{Re}_x}},\ \frac{\theta}{x}=\frac{0.664}{\sqrt{\mathrm{Re}_x}}$$ (Re가 증가하면 경계층 두께는 얇아진다)

$$\delta_1=\int_0^{\delta}\left(1-\frac{u}{U_0}\right)dy$$

9.2.2 Prandtl/Blasius Exact Solution

Note:

$$\eta(x,y)=\frac{y}{\delta} \rightarrow dy=\delta d\eta$$

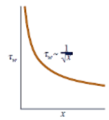
$$\eta(x,y)=\frac{y}{\delta}=y\sqrt{\frac{U_0}{\nu x}},\quad f(\eta)=\frac{\psi}{\sqrt{U_0 \nu x}},\quad f'(\eta)=\frac{u}{U_0}$$

$$\delta^*=\int_0^{\delta}\left(1-\frac{u}{U_0}\right)dy=\frac{\partial y}{\partial \eta}\int_0^1 (1-f'(\eta))d\eta=1.7208\sqrt{\frac{\nu x}{U_0}}$$

$$\theta=\int_0^{\delta}\frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy=\frac{\partial y}{\partial \eta}\int_0^1 f'(\eta)(1-f'(\eta))d\eta=0.664\sqrt{\frac{\nu x}{U_0}}$$

✓속도분포를 이용한 벽면전단응력(τ_w) 계산

$$\tau_w=0.332U^{3/2}\sqrt{\frac{\rho\mu}{x}}$$

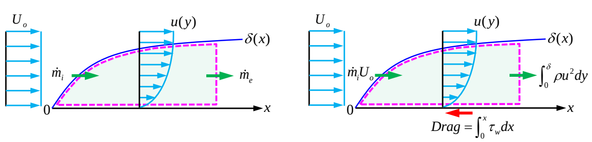


Approximate Solution of BL Eq.
(Momentum Integral Equation proposed by Von Karman)

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq 0$	$dp/dx=0$	$dp/dx\neq 0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIE)	O	O	O	O

Momentum Integral Equation of BL

✓**In the case of U_0=constant** (**without pressure gradient effect, $dp/dx=0$**)



(a) Mass conservation (b) Momentum conservation

1) Continuity eq. : $0=\cancel{\frac{\partial}{\partial t}\int_{cv}\rho dV}+\int_{cs}\rho\vec{V}\cdot d\vec{A}$

assumptions : ① **steady**($\partial/\partial t=0$), ② **2-D flow**

$$\therefore \dot{m}_i=\dot{m}_e=\int_0^{\delta}\rho u dy$$

Momentum Integral Equation of BL

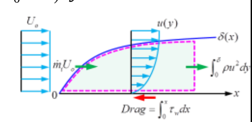
2) Momentum eq. : $\sum F_x=F_{Sx}+\cancel{F_{Bx}}=\cancel{\frac{\partial}{\partial t}}\int_{cv}u\rho dV+\int_{cs}u\rho\vec{V}\cdot d\vec{A}$

assumptions : ③ **No body force** (f_x=0), ④ **free stream with U_0=Constant**
→ No consider, $(-dp/dx=0)$

$$\sum F_x=F_{Sx}=\int_{cs}u\rho\vec{V}\cdot d\vec{A} \quad \text{(for unit width=1)}$$

$$-\int_0^x \tau_w dx=-\underbrace{\int_0^{\delta}\rho u U_0 dy}_{\dot{m}_i U_o=U_o\int_0^{\delta}\rho u dy}+\underbrace{\int_0^{\delta}\rho u u dy}_{\dot{m}_e u=\int_0^{\delta}\rho u u dy}=-\int_0^{\delta}\rho u(U_0-u)dy$$

Inflow into CV with U_0 Outflow from CV with u



(b) Momentum conservation

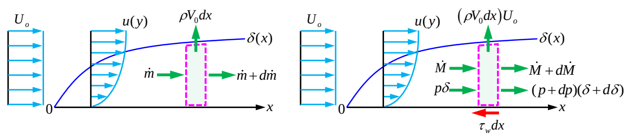
differentiate both sides by x (because of $\frac{dU_0}{dx}=0$)

$$\tau_w=\frac{d}{dx}\int_0^{\delta}\rho u(U_0-u)dy=\boxed{\rho U_0^{\,2}\frac{d}{dx}\int_0^{\delta}\frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy=\rho U_0^{\,2}\frac{d}{dx}\delta_m}$$

Momentum Integral Equation of BL

✓**In the caser of $U_0 = U_0(x)$ (with pressure gradient effect, $dp/dx \neq 0$)**

Note: Boundary layer is not streamline, so the flow is present through BL.



(a) Mass conservation **(b) Momentum conservation**

1) In the outer region where no viscosity effect is considered,

Bernoulli eq. : $\frac{1}{2}\rho U_0^2 + p + \cancel{\rho g h} = H(\text{const.})$ Here, h is height from base line

in the case of free stream, δ is in disregard of h ($\because L >> \delta$) (boundary layer is too thin !!)

Differentiate above formula by x : $-\frac{dp}{dx} = \rho U_0 \frac{dU_0}{dx}$ (comparing with U_0=constant)

(force due to $\partial p/\partial x$ does not appear in the case of free stream with U_0=constant)

Momentum Integral Equation of BL

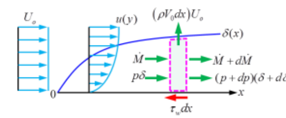
2) Continuity eq. :

$$0 = \cancel{\frac{\partial}{\partial t}\int_{cv} \rho dV} + \int_{cs}\rho \vec{V}\cdot d\vec{A} \rightarrow 0 = \int_{cs} \rho\vec{V}\cdot d\vec{A}$$

$$0 = -\dot m + (\dot m + d\dot m) + \rho V_0 dx$$

$$0 = d\dot m + \rho V_0 dx \rightarrow 0 = \frac{d}{dx}\int_0^\delta \rho u dy + \rho V_0$$

$$\rho V_0 = -\frac{d}{dx}\int_0^\delta \rho u dy$$



(b) Momentum conservation

3) Momentum eq. (M=momentum)

$$\sum F_x = F_{Sx} = \int_{cs} u\rho \vec{V}\cdot d\vec{A}$$

$$p\delta - (p+dp)(\delta + d\delta) - \tau_w dx = -\dot M + (\dot M + d\dot M) + \underbrace{(\rho V_0 dx)}_{\dot m} U_0$$

$$p\delta - p\delta - \cancel{p d\delta} - \delta dp - \cancel{dp d\delta} - \tau_w dx = d\dot M + (\rho V_0 dx)U_0$$

$$-\delta dp - \tau_w dx = d\dot M + \rho V_0 dx U_0$$

(here, $p d\delta << \delta dp$)

Momentum Integral Equation of BL

$$-\delta dp - \tau_w dx = d\dot M + \rho V_0 dx U_0$$

$$-\delta\frac{dp}{dx} - \tau_w = \frac{d}{dx}\dot M + \rho V_0 U_0$$

$$-\delta\left(-\rho U_0 \frac{dU_0}{dx}\right) - \tau_w = \frac{d}{dx}\int_0^\delta \rho u^2 dy - U_0\frac{d}{dx}\int_0^\delta \rho u dy$$

$$\tau_w = U_0\frac{d}{dx}\int_0^\delta \rho u dy - \frac{d}{dx}\int_0^\delta\rho u^2 dy + \delta\rho U_0\frac{dU_0}{dx}$$

Here, since $U_0=U_0(x)$ is as the function of x

$$\left\{ \begin{aligned} \frac{d}{dx}\int_0^\delta \rho u U_0 dy &= \frac{d}{dx}\left(U_0\int_0^\delta \rho u dy\right) \\ &= \frac{dU_0}{dx}\int_0^\delta\rho u dy + U_0\frac{d}{dx}\int_0^\delta \rho u dy \end{aligned} \right\}$$

Momentum Integral Equation of BL

$$\tau_w = \frac{d}{dx}\int_0^\delta \rho u U_0 dy - \frac{dU_0}{dx}\int_0^\delta \rho u dy - \frac{d}{dx}\int_0^\delta\rho u^2 dy + \delta\rho U_0\frac{dU_0}{dx}$$

$$= \frac{d}{dx}\int_0^\delta \rho u(U_0 - u)dy - \frac{dU_0}{dx}\int_0^\delta \rho u dy + \delta\rho U_0\frac{dU_0}{dx}$$

Here, $\int_0^\delta \rho U_0 dy = \rho U_0\delta$

$$= \frac{d}{dx}\int_0^\delta\rho u(U_0-u)dy + \frac{dU_0}{dx}\int_0^\delta \rho(U_0-u)dy$$

Using the displacement thickness (δ_d) and momentum thickness (δ_m)

$$= \frac{d}{dx}\rho U_0^2 \underbrace{\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy}_{\delta_m} + \frac{dU_0}{dx}\rho U_0 \underbrace{\int_0^\delta\left(1-\frac{u}{U_0}\right)dy}_{\delta_d}$$

$$= \frac{d}{dx}\rho U_0^2\delta_m + \frac{dU_0}{dx}\rho U_0\delta_d$$

δ_d : Displacement thickness
δ_m : Momentum thickness

Momentum Integral Equation of BL

✓**Summary of MIE**

1) without pressure gradient effect, *dp/dx*=0)

$$\tau_w = \frac{d}{dx}\int_0^\delta \rho u\left(U_0 - u\right)dy \;\left(U_o = \text{Constant}\right)$$

$$= \boxed{\rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy} = \rho U_0^{\,2}\frac{d}{dx}\delta_m$$

2) with pressure gradient effect, *dp/dx*=0)

$$\tau_w = \boxed{\rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy + \rho U_0\frac{d}{dx}\int_0^\delta\left(1-\frac{u}{U_0}\right)dy} \;\left(U_o \neq \text{Constant}\right)$$

$$= \rho U_0^{\,2}\frac{d}{dx}\delta_m \qquad + \rho U_0\frac{dU_0}{dx}\delta_d$$

Unlike the Blasius solution, which is exact, approximate solution method assumes an **approximate shape of the velocity profile.** This velocity profile is then utilized to evaluate quantities related to the governing differential equation, given below by Karman and Pohlhausen. This method, which is called the momentum integral method, changes the two equations given by Prandtl into a single differential equation. Integral momentum eq. can be applied in **both laminar and turbulent flow**, because any assumption was applied.

Laminar Boundary Layer

without pressure gradient(dp/dx=0)

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq0$	$dp/dx=0$	$dp/dx\neq0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIE)	O	O	O	O

Application of MIE-Laminar boundary layer

1) Linear velocity distribution using momentum integral eq.

• **Velocity distribution:** $\dfrac{u(y)}{U_0} = \eta = \dfrac{y}{\delta(x)} \rightarrow u(y) = \dfrac{U_0}{\delta(x)}y$

constant at fixed x

$$\tau_w = \frac{d}{dx}\int_0^\delta \rho u\left(U_0-u\right)dy + \frac{dU_0}{dx}\int_0^\delta \rho\left(U_0-u\right)dy$$

$$= \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy = \rho U_0^{\,2}\frac{d}{dx}\delta_m$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg)_{y=0}$$

$$= \left[\mu U_0\frac{\partial}{\partial y}\left(\frac{y}{\delta}\right)\right]_{y=0} = \left[\mu\frac{U_0}{\delta}\right] \rightarrow$$

substitute this eq. into momentum integral eq.

$$\mu\frac{U_0}{\delta} = \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)dy = \rho U_0^{\,2}\frac{d}{dx}\left[\int_0^\delta \frac{y}{\delta}dy - \int_0^\delta \frac{y^2}{\delta^2}dy\right]$$

$$= \rho U_0^{\,2}\frac{d}{dx}\left(\frac{\delta^2}{2\delta}-\frac{\delta^3}{3\delta^2}\right) = \rho U_0^{\,2}\frac{d}{dx}\left(\frac{\delta}{6}\right)$$

$$= \frac{\rho U_0^{\,2}}{6}\frac{d\delta}{dx}$$

Application of MIE-Laminar boundary layer

$$\mu U_0 = \frac{\rho U_0^{\,2}}{6}\delta\frac{d\delta}{dx} = \frac{\rho U_0^{\,2}}{6}\frac{d}{dx}\left(\frac{\delta^2}{2}\right) \qquad \text{Here, } \frac{d}{dx}\left(\frac{\delta^2}{2}\right) = \delta\frac{d\delta}{dx}$$

$$\frac{d}{dx}\left(\frac{\delta^2}{2}\right) = \mu U_0\frac{6}{\rho U_0^{\,2}} = \frac{\mu}{\rho}\frac{6}{U_0} = \frac{\nu\times 6}{U_0}$$

Solving the DE using BC : $\delta(0)=0$

$$\frac{d}{dx}\left(\frac{\delta^2}{2}\right) = \frac{6\nu}{U_0}$$

$$\frac{\delta^2}{2} = \frac{6\nu}{U_0}x$$

$$\delta^2 = \frac{12\nu x}{U_0} = \frac{12\nu x^2}{U_0 x}$$

$$\frac{\delta}{x} = \sqrt{\frac{12\nu}{U_0 x}} = \frac{3.46}{\text{Re}_x^{\frac{1}{2}}}$$

The skin friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^{\,2}} = \frac{\mu\frac{U_0}{\delta}}{\frac{1}{2}\rho U_0^{\,2}} = \frac{2\nu}{U_0\delta}$$

$$= \frac{2\nu}{U_0}\frac{1}{x}\frac{x}{\delta} = \frac{2}{\text{Re}_x}\frac{\text{Re}_x^{\frac{1}{2}}}{\sqrt{12}} = \frac{0.577}{\text{Re}_x^{\frac{1}{2}}}$$

For linear velocity profile

$$\boxed{\frac{\delta}{x} = \frac{3.46}{\text{Re}_x^{\frac{1}{2}}} \quad C_f = \frac{0.577}{\text{Re}_x^{\frac{1}{2}}}}$$

Application of MIE-Laminar boundary layer

Mean fanning friction coefficient ($\bar{C}_f$) **for** $0 \le x \le L$

$$\bar{C}_f = \frac{1}{L}\int_0^L C_f\,dx = \frac{0.577}{L}\int_0^L Re_x^{-0.5}\,dx = \frac{0.577}{L}\sqrt{\frac{\nu}{U_0}}\int_0^L \frac{1}{\sqrt{x}}dx = \frac{0.577}{L}\sqrt{\frac{\nu}{U_0}}\,2L^{0.5}$$

$$= \frac{1.154}{Re^{1/2}}$$

$$= 2C_f$$

For linear velocity profile

$$\frac{\delta}{x} = \frac{3.46}{Re_x^{\frac{1}{2}}} \qquad C_f = \frac{0.577}{Re_x^{\frac{1}{2}}}$$

Wall shear stress, τ_w

$$\tau_w = \mu\frac{u_0}{\delta} = \mu U_0\frac{Re_x^{\frac{1}{2}}}{3.46x} = 0.289\mu U_0 x^{-1}\sqrt{\frac{U_0 x}{\nu}} = 0.289\frac{\mu}{\rho}\frac{U_0^2}{U_0}\rho x^{-1}\sqrt{\frac{U_0 x}{\nu}}$$

$$= 0.289\sqrt{\frac{\nu}{U_0 x}}$$

$$= 0.289\,Re_x^{-\frac{1}{2}}$$

Application of MIE-Laminar boundary layer

2) Quadratic velocity distribution

- **Velocity distribution:** $u(y)=a+by+cy^2$

$$BC:\begin{cases}(1)\ u=v=0 & \text{at } y=0\\ (2)\ u=U_0,\ \boxed{\dfrac{\partial u}{\partial y}=0 \ \text{ at } y=\delta} \end{cases}$$

(no shear stress)

Note that at the edge of the defined edge of the boundary layer $u=0.99U$ and $du/dy=0$. However we approximate them with the rounded values. This is the reason why the solution method by Momentum Integral Method is considered an approximate one.

By ①: $0 = a+0+0 \rightarrow a=0$

By ②: $u = by+cy^2,\ 0=b+2c\delta$

$$U_0 = b\delta + c\delta^2 = (-2c\delta)\delta + c\delta^2 = -2c\delta^2 + c\delta^2 = -c\delta^2 \ \rightarrow\ c = -\frac{U_0}{\delta^2}$$

$$\frac{\partial u}{\partial y} = b+2cy \ \rightarrow\ 0=b+2c\delta \ \rightarrow\ b=-2c\delta = -2\left(-\frac{U_0}{\delta^2}\right)\delta = \frac{2U_0}{\delta}$$

$$u(y) = by+cy^2 = \frac{2U_0}{\delta}y - \frac{U_0}{\delta^2}y^2 = U_0\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)$$

$$\frac{u}{U_0} = 2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$$ **: Parabolic velocity profile**

Remember the use of the boundary layer velocity profile is only meaningful when $0 \le y \le \delta$.

Application of MIE-Laminar boundary layer

Introduce non-dimensional parameter : $\eta = \dfrac{y}{\delta}\ (\delta\eta = y \ \rightarrow\ \delta d\eta = dy)$

- **Velocity distribution :** $\dfrac{u}{U_0} = 2\left(\dfrac{y}{\delta}\right)-\left(\dfrac{y}{\delta}\right)^2 = 2\eta-\eta^2$

$$BC:\begin{cases}(1)\ \eta=0 & \text{at } y=0\\ (2)\ \eta=1 & \text{at } y=\delta\end{cases} \qquad \left(\eta=\frac{y}{\delta}=\frac{\delta}{\delta}=1\right)$$

- **Wall shear stress,** τ_w

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right)_{y=0}, \qquad u=u(y)=u(\eta),\ \eta=\eta(y)$$

$$= \mu\left.\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)_{\eta=0}$$

$$= \mu\left.\frac{1}{\delta}\frac{\partial u}{\partial \eta}\right)_{\eta=0} = \mu\frac{1}{\delta}U_0(2-2\eta)_{\eta=0} = 2\frac{\mu}{\delta}U_0$$

Application of MIE-Laminar boundary layer

Using the above results in the momentum integral equation for flat plate:

$$\tau_w = \rho U_0^2\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy = \rho U_0^2\frac{d}{dx}\delta_m \ \left(\because \frac{\partial U_0}{\partial x}=0\right)$$

$$= \rho U_0^2\frac{d}{dx}\int_0^1 \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)\delta d\eta$$

$$\left(\delta\eta = y,\ \delta d\eta = dy,\ y=\delta \ \rightarrow\ \eta=\frac{\delta}{\delta}=1\right)$$

$$= \rho U_0^2\frac{d}{dx}\delta_m$$

$$= \rho U_0^2\frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

$$\delta_m = \int_0^1 \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)\delta d\eta$$

$$= \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\delta d\eta$$

$$= \int_0^1 (2\eta-5\eta^2+4\eta^3-\eta^4)\delta d\eta$$

$$= \left[\eta^2-\frac{5}{3}\eta^3+\eta^4-\frac{1}{5}\eta^5\right]_0^1 \delta d\eta = \frac{2}{15}\delta$$

Application of MIE-Laminar boundary layer

Combining two wall shear stress

$$2\frac{\mu}{\delta}U_0 = \rho U_0^{\ 2}\frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

$$\frac{2\mu U_0 15}{\rho U_0^{\ 2}2} = \delta\frac{d\delta}{dx} = \frac{1}{2}\frac{d\delta^2}{dx}$$

$$\frac{d\delta^2}{dx} = \frac{30\nu}{u_0} \rightarrow \delta^2 = \frac{30\nu}{u_0}x$$

$$\delta = \sqrt{\frac{30\nu x}{U_0}} = 5.48x\sqrt{\frac{\nu}{U_0 x}} = \frac{5.48x}{\mathrm{Re}_x^{\frac{1}{2}}}$$

$$\frac{\delta}{x} = \frac{5.48}{\mathrm{Re}_x^{\frac{1}{2}}}$$

Application of MIE-Laminar boundary layer

$$\tau_w = 2\frac{\mu}{\delta}U_0 = 2\mu U_0\frac{1}{\delta} = 2\mu U_0\frac{\mathrm{Re}_x^{\ 0.5}}{5.48x} = \frac{2}{5.48}\frac{U_0^{\ 2}}{U_0}\frac{\mu}{\rho}\rho x^{-1}\sqrt{\frac{U_0 x}{\nu}}$$

$$= 0.365\rho U_0^{\ 2}\sqrt{\frac{\nu}{U_0 x}} = \boxed{0.365\rho U_0^{\ 2}\,\mathrm{Re}_x^{\ -0.5}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^{\ 2}} = \frac{0.365\rho U_0^{\ 2}}{\frac{1}{2}\rho U_0^{\ 2}\sqrt{\mathrm{Re}_x}} = \boxed{\frac{0.730}{\sqrt{\mathrm{Re}_x}}}$$

$$\overline{C}_f = \frac{1}{L}\int_0^L C_f\,dx = \frac{0.730}{L}\int_0^L \mathrm{Re}_x^{\ -0.5}\,dx = \frac{0.730}{L}\sqrt{\frac{\nu}{U_0 x}}dx$$

$$= \frac{0.730}{L}\sqrt{\frac{\nu}{U_0}}\int_0^L\frac{1}{\sqrt{x}}dx = \frac{0.730}{L}\sqrt{\frac{\nu}{U_0}}2L^{0.5} = 1.46\sqrt{\frac{\nu L}{U_0 L^2}} = 1.46\sqrt{\frac{\nu}{U_0 L}}$$

$$= \boxed{1.46\,\mathrm{Re}_L^{\ -0.5}}$$

Application of MIE-Laminar boundary layer

3) Sinusoidal velocity distribution (U_o=C, $dx/dp=0$)

- **Velocity distribution:** $\frac{u(y)}{U_0} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right),\quad \frac{u}{U_0} = 1 \text{ at } y > \delta$

Substituting given velocity distribution into Momentum Integral Equation:

$$\tau_w = \rho U_0^{\ 2}\frac{d}{dx}\int_0^\delta\frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy,\ \left(\eta = \frac{y}{\delta} \rightarrow dy = \delta d\eta\right)$$

$$= \rho U_0^{\ 2}\frac{d\delta}{dx}\int_0^1\sin\left(\frac{\pi}{2}\right)\eta\left(1-\sin\left(\frac{\pi}{2}\right)\right)d\eta,\ \left(y=\delta \rightarrow \eta = \frac{y}{\delta} = \frac{\delta}{\delta} = 1\right)$$

$$= \rho U_0^{\ 2}\frac{d\delta}{dx}\int_0^1\left(\sin\left(\frac{\pi}{2}\right)\eta - \sin^2\left(\frac{\pi}{2}\right)\eta\right)d\eta$$

$$= \rho U_0^{\ 2}\frac{d\delta}{dx}\left(\frac{2}{\pi}\right)\left[-\cos\left(\frac{\pi}{2}\right)\eta - \frac{1}{2}\left(\frac{\pi}{2}\right)\eta + \frac{1}{4}\sin\pi\eta\right]_0^1$$

$$= \rho U_0^{\ 2}\frac{d\delta}{dx}\left(\frac{2}{\pi}\right)\left[0+1-\frac{\pi}{4}+0+0-0\right]_0^1 = 0.137\rho U_0^{\ 2}\frac{d\delta}{dx}$$

Application of MIE-Laminar boundary layer

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right)_{y=0},\qquad u = u(y) = u(\eta),\ \eta = \eta(y)\quad \frac{u}{U_0} = \sin\left(\frac{\pi}{2}\eta\right),\ \eta = \frac{y}{\delta}$$

$$= \mu\left.\frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}\right)_{\eta=0}$$

$$= \mu\left.\frac{1}{\delta}\frac{\partial u}{\partial\eta}\right)_{\eta=0} = \mu\frac{1}{\delta}\frac{\partial}{\partial\eta}\left.\left(U_0\sin\left(\frac{\pi}{2}\eta\right)\right)\right)_{\eta=0} = \mu\frac{U_0}{\delta}\frac{\pi}{2}\left.\cos\left(\frac{\pi}{2}\eta\right)\right)_{\eta=0} = \frac{\pi\mu U_0}{2\delta}$$

Therefore, $\tau_w = \frac{\pi\mu U_0}{2\delta} = 0.137\rho U_o^2\frac{d\delta}{dx}$

Separating variables and then Integrating, we obtain:

$$\delta d\delta = 11.5\frac{\mu}{\rho U_0}dx$$

$$\frac{\delta^2}{2} = 11.5\frac{\mu}{\rho U_0}x + C = 11.5\frac{\mu}{\rho U_0}x\ \ (C=0,\ \because x=0,\ \delta=0)$$

Application of MIE-Laminar boundary layer

Growth of the boundary layer:

$$\delta = \sqrt{20.3\frac{\mu x}{\rho U_0}} \quad or \quad \frac{\delta}{x} = 4.80\sqrt{\frac{\mu}{\rho U_0 x}} = \frac{4.80}{\sqrt{\mathrm{Re}_x}}$$

Wall shear stress:

$$\tau_w = \frac{\pi\mu U_0}{2\delta} = \frac{\pi\mu U_0}{2}\frac{1}{\delta} = \frac{\pi\mu U_0}{2}\frac{\mathrm{Re}_x^{0.5}}{4.80x} = \frac{\pi\mu U_0}{9.60x}\sqrt{\frac{\rho U_0 x}{\mu}} = \frac{\pi\rho U_o^2}{9.60}\sqrt{\frac{\mu}{\rho U_0 x}}$$
$$= 0.327\rho U_o^2\,\mathrm{Re}_x^{-0.5}$$

Skin friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^{\,2}} = \frac{0.327\rho U_0^{\,2}}{\frac{1}{2}\rho U_0^{\,2}\sqrt{\mathrm{Re}_x}} = \frac{0.654}{\sqrt{\mathrm{Re}_x}}$$

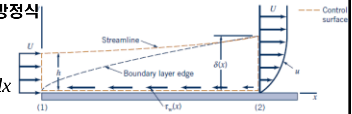
$$\overline{C}_f = \frac{1}{L}\int_0^L C_f\,dx = \frac{0.654}{L}\int_0^L \mathrm{Re}_x^{-0.5}\,dx = \frac{0.654}{L}\sqrt{\frac{\nu}{U_0 x}}dx = \frac{0.654}{L}\sqrt{\frac{\nu}{U_0}}\int_0^L \frac{1}{\sqrt{x}}dx$$
$$= \frac{0.654}{L}\sqrt{\frac{\nu}{U_0}}2L^{0.5} = 1.308\sqrt{\frac{\nu L}{U_0 L^2}} = 1.308\sqrt{\frac{\nu}{U_0 L}} = 1.308\,\mathrm{Re}_L^{-0.5}$$

9.2.3 Momentum Integral BL Eq.

✓**평행 평판에 대한 운동량 적분 경계층방정식 (근사해)**

i) 압력구배가 없는 운동량적분방정식:

검사체적 내 정상유동에 대한 x 방향 운동량방정식

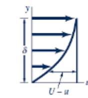


$$\sum F_x = \rho\int_{(1)} u\vec{V}\cdot\vec{n}dA + \rho\int_{(2)} u\vec{V}\cdot\vec{n}dA$$

$$\sum F_x = -D = -\int_{\text{plate}}\tau_w dA = -b\int_{\text{plate}}\tau_w dx$$

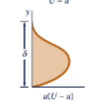
$$-D = \rho\int_{(1)} U(-U)dA + \rho\int_{(2)} u^2 dA \;\rightarrow\; D = \rho U^2 bh - \rho b\int_0^{\delta} u^2 dA$$

질량보존법칙에 의해 단면 (1)을 지나는 유량과 단면(2)를 지나는 유량이 같으므로



$$Uh = \int_0^{\delta} u dy \;\rightarrow\; \rho U^2 bh = \rho b\int_0^{\delta} U u dy$$

운동량 방정식과 질량보존법칙 결과를 결합하면, 운동량 결손(deficit)으로 표현됨



$$D = \rho b\int_0^{\delta} u(U-u)dy = \rho b U^2\theta$$

$$\frac{dD}{dx} = \rho b U^2\frac{d\theta}{dx} = b\tau_w \;\rightarrow\; \boxed{\tau_w = \rho U^2\frac{d\theta}{dx}}$$ **운동량적분방정식**

9.2.7 Momentum Integral BL Eq.

ii) 압력구배가 있는 운동량 적분경계층방정식(With Nonzero Pressure Gradient)

압력구배가 없는 운동량적분방정식:

$$\tau_w = \rho U^2\frac{d\theta}{dx}$$

압력구배가 있는 운동량적분방정식:

$$\frac{dp}{dx} = -\rho U_{fs}\frac{dU_{fs}}{dx}$$

$$\tau_w = \rho\frac{d}{dx}\left(U_{fs}^2\theta\right) + \rho\delta^* U_{fs}\frac{dU_{fs}}{dx}$$

$$\frac{p}{\rho} + \frac{1}{2}U^2 = C$$

$$\frac{1}{\rho}\frac{dp}{dx} + U\frac{dU}{dx} = 0 \;\rightarrow\; \frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx}$$

9.2.3 Momentum Integral BL Eq.

✓**속도분포 변화에 따른 경계층 방정식의 근사해 계산**

1) 일반적인 속도분포 함수

$$\frac{u}{U} = g(Y) \text{ for } 0 \le Y \le 1, \text{ and } \frac{u}{U} = 1 \text{ for } Y > 1 \leftarrow BC: g(0) = 0 \text{ and } g(1) = 1$$

$$D = \rho b\int_0^{\delta} u(U-u)dy = \rho b U^2\delta\int_0^1 g(Y)\left[1-g(Y)\right]dY = \rho b U^2\delta C_1$$

$$C_1 = \int_0^1 g(Y)\left[1-g(Y)\right]dY$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu U}{\delta}\frac{dg}{dY}\bigg|_{Y=0} = \frac{\mu U}{\delta}C_2 \;\rightarrow\; C_2 = \frac{dg}{dY}\bigg|_{Y=0}$$

$$\delta d\delta = \frac{\mu C_2}{\rho U C_1}dx$$

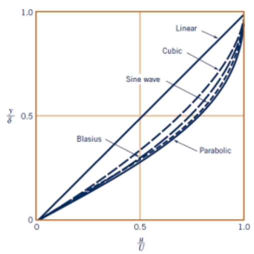
$$\delta = \sqrt{\frac{2\nu C_2 x}{U C_1}} \quad \text{or} \quad \frac{\delta}{x} = \frac{\sqrt{2C_2/C_1}}{\sqrt{\mathrm{Re}_x}}$$

$$\tau_w = \sqrt{\frac{C_1 C_2}{2}}U^{3/2}\sqrt{\frac{\rho\mu}{x}}$$

9.2.3 Momentum Integral BL Eq.

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \rightarrow c_f = \sqrt{2C_1C_2}\sqrt{\frac{\mu}{\rho U x}} = \frac{\sqrt{2C_1C_2}}{\sqrt{\mathrm{Re}_x}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_{Df} = \frac{D_f}{\frac{1}{2}\rho U^2 b\ell} = \frac{b\int_0^\ell \tau_w dx}{\frac{1}{2}\rho U^2 b\ell} = \frac{1}{\ell}\int_0^\ell c_f dx \rightarrow C_{Df} = \frac{\sqrt{8C_1C_2}}{\sqrt{\mathrm{Re}_\ell}} = \frac{1.328}{\sqrt{\mathrm{Re}_\ell}}$$



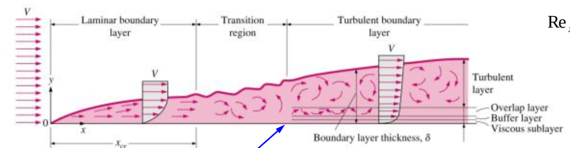
Flat Plate Momentum Integral Results for Various Assumed Laminar Flow Velocity Profiles

Profile Character	$\delta \mathrm{Re}_x^{1/2}/x$	$c_f \mathrm{Re}_x^{1/2}$	$C_{Df} \mathrm{Re}_\ell^{1/2}$
a. Blasius solution	5.00	0.664	1.328
b. Linear $u/U = y/\delta$	3.46	0.578	1.156
c. Parabolic $u/U = 2y/\delta - (y/\delta)^2$	5.48	0.730	1.460
d. Cubic $u/U = 3(y/\delta)/2 - (y/\delta)^3/2$	4.64	0.646	1.292
e. Sine wave $u/U = \sin[\pi(y/\delta)/2]$	4.79	0.655	1.310

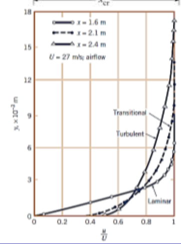
9.2.4 Transition from L. to Turbulent

✓층류 경계층에서 난류 경계층으로 천이(Transition)

층류에서 난류 경계층으로의 천이 과정은 Reynolds number의 함수(임계 Re. No.)



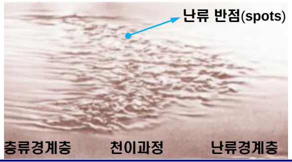
$$\mathrm{Re}_{x,cr} = \frac{\rho V x}{\mu}$$



Onset of Turbulent Boundary Layer

Critical Reynolds Number: 표면조도, 곡률 등 복잡한 함수

$$\mathrm{Re}_{x,cr} \approx 2\times10^5 \sim 3\times10^6 \rightarrow \mathrm{Re}_x = 5\times10^5 \text{ for flat plate}$$



난류 파열(burst)

난류 반점(spots) 생성

난류 반점(spots) 성장

난류 반점(spots) 확산

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Turbulent Boundary Layer

without pressure gradient(dp/dx=0)

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq0$	$dp/dx=0$	$dp/dx\neq0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIE)	O	O	O	O

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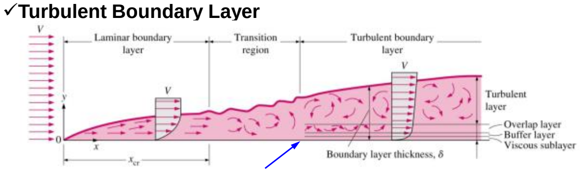
Application of MIE-Turbulent boundary layer

✓Turbulent Boundary Layer



Onset of Turbulent Boundary Layer

Critical Reynolds Number:

$\mathrm{Re}_x \geq 5\times10^5$ for flat plate

$\mathrm{Re}_x \geq 2 \sim 4\times10^5$ for sphere or cirular cylinder

We know that turbulent flow occurs if the flow velocity is large enough (or, viscosity is small enough) to create a Reynolds number greater than the critical Reynolds number over an object. For spheres or circular cylinders this **critical Reynolds number is between 2 to 4x10⁵. For flat plate flows this is around 500,000.**

It is important to recognize that turbulent flows have two components: (i) a mean($\bar{u}$), and (ii) a random one(u'). The random u' cannot be determined without statistical means. Therefore for turbulent fluid flows, we usually work with a time averaged mean flow $\bar{u}$.

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Application of MIE-Turbulent boundary layer

✓**Turbulent Flow Velocity Distribution :**

Turbulent flows in boundary layers over flat plates may be represented by the **power law velocity profile**:

$$\frac{u}{u_*} = 8.75\left(\frac{yu_*}{\nu}\right)^{\frac{1}{7}} \quad (\text{one} - \text{seventh power law})$$

$$u^+ = 8.75\left(y^+\right)^{\frac{1}{7}}$$

You understand that whenever we speak about turbulent flows here, we are representing the mean flow.
(난류는 파동속도에 의해 정의되지만, 여기서는 평균속도를 고려하자)

If $y = \delta, \quad u = U_0$

$$\frac{U_0}{u_*} = 8.75\left(\frac{\delta u_*}{\nu}\right)^{\frac{1}{7}}$$

$$\frac{\dfrac{u}{u_*}}{\dfrac{U_0}{u_*}} = \frac{8.75\left(\dfrac{yu_*}{\nu}\right)^{\frac{1}{7}}}{8.75\left(\dfrac{\delta u_*}{\nu}\right)^{\frac{1}{7}}} \rightarrow \frac{u}{U_0} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

$$u = U_0\left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

$$\frac{du}{dy} = \frac{U_0}{7}\left(\frac{y}{\delta}\right)^{-\frac{6}{7}} \rightarrow \left.\frac{du}{dy}\right)_{y=0} = \frac{U_0}{7}\left(\frac{y}{\delta}\right)^{-\frac{6}{7}} = \infty$$

This velocity profile is widely used in turbulent BL since the equation can be represent as a function of $\eta = y/\delta$ even though the profile can not apply to estimate the wall shear stress at $y=0$

Application of MIE-Turbulent boundary layer

Substituting this result into momentum integral equation for without pressure gradient (*dp/dx*=0)

$$\tau_w = \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy = \rho U_0^{\,2}\frac{d}{dx}\delta_m$$

$$\tau_w = \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy + \rho U_0\frac{d}{dx}\int_0^\delta\left(1-\frac{u}{U_0}\right)dy$$

$$= \rho U_0^{\,2}\frac{d}{dx}\delta_m \qquad + \rho U_0\frac{dU_0}{dx}\delta_d$$

$$\tau_w = \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta \frac{u}{U_0}\left(1-\frac{u}{U_0}\right)dy = \rho U_0^{\,2}\frac{d}{dx}\delta_m$$

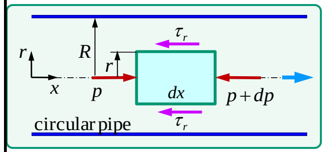
$$= \rho U_0^{\,2}\frac{d}{dx}\int_0^\delta\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\left(1-\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right)dy = \rho U_0^{\,2}\frac{d}{dx}\left(\int_0^\delta\left(\frac{y}{\delta}\right)^{\frac{1}{7}}dy - \int_0^\delta\left(\frac{y}{\delta}\right)^{\frac{2}{7}}dy\right)$$

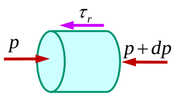
$$= \frac{7}{72}\rho U_0^{\,2}\left(\frac{d\delta}{dx}\right)$$

Both τ_w & δ → unknown parameters
Relations of τ_w & δ is needed.
(**난류이므로 벽면 전단응력을 구하는 공식이 없음**)

This profile covers a fairly broad range of turbulent Reynolds number for 6 < ***n*** < 10. The most popular one is ***n***=7. Although this velocity profile is an excellent representation of the real turbulent flow, **this may not be used to calculate skin friction coefficient** in the approximate solution method seen earlier, since $\partial u/\partial y)_{y=0}$ **will be negligible for this profile**. For the purpose of calculating shear stress we use an **experimental results**:(**실험결과를 이용해야 함**)

Application of MIE-Turbulent boundary layer





Assumptions

① steady flow
② ΣF_{Bx}=0
③ A=Constant (V_1=V_2)
④ Fully development

$$\sum \vec{F}_x = \cancel{\vec{F}_{Bx}} + \vec{F}_{Sx} = \cancel{\frac{\partial}{\partial t}}\int_{CV}\vec{V}\rho d\forall + \int_{CS}\vec{V}\rho\left(\vec{V}\cdot d\vec{A}\right) = 0$$

$$\vec{F}_{Sx} = p\pi r^2 - (p+dp)\pi r^2 - \tau_r 2\pi r dx = 0$$

$$-dp\pi r^2 = \tau_r 2\pi r dx$$

$$\tau_r = -\frac{r}{2}\frac{dp}{dx}$$

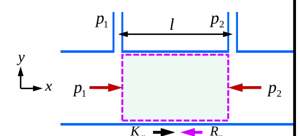
$$\boxed{\tau_{r=R} = \tau_w = -\frac{R}{2}\frac{dp}{dx}}$$

This formula is the relation connecting the wall shear stress with flow direction pressure gradient. In the derivation process, the momentum equation was only used without the assumption of shear stress and velocity field (Namely, not apply the Newton's viscous law. Therefore, this equation may be applied both **laminar and turbulent flow**.

Application of MIE-Turbulent boundary layer

Relation of wall shear stress and flow direction pressure gradient

$$\tau_w = -\frac{R}{2}\frac{dp}{dx} = \frac{R}{2}\frac{\Delta p}{l}, \quad (-\Delta p = p_1 - p_2)$$



$$(\Delta p = p_2 - p_1) \rightarrow (-\Delta p = p_1 - p_2)$$

From the Darcy-Weisbach equation

$$h = \lambda\frac{l}{d}\frac{V^2}{2g} \quad (\text{Darcy} - \text{Weisbach equation})$$

$$\Delta p = \gamma h = \lambda\frac{l}{d}\frac{V^2}{2g}\gamma = \lambda\frac{l}{d}\frac{\rho V^2}{2}$$

Combining two relations, we obtain:

$$\tau_w 2\frac{l}{R} = \lambda\frac{l}{D}\frac{\rho V^2}{2}$$

$$\tau_w = \lambda\frac{R}{2D}\frac{\rho V^2}{2} = \lambda\frac{R}{4R}\frac{\rho V^2}{2} = \frac{\lambda}{4}\frac{\rho V^2}{2} = \boxed{\frac{1}{8}\lambda\rho V^2}$$

λ : friction factor
V : mean velocity

Application of MIE-Turbulent boundary layer

For the smooth circular pipe, relationship between λ and Reynolds Number:

$$\lambda = \frac{0.316}{\mathrm{Re}_D^{1/4}} \quad \text{for} \quad 3\times10^3 \langle \mathrm{Re} \langle 10^5 \quad (\text{Blasius' s equation}) \rightarrow$$ **See next slide**

Substituting this equation into τ_w in the fully developed turbulent flow we obtain:

$$\tau_w = \frac{1}{8}\lambda\rho V^2 = \frac{1}{8}\left(\frac{0.316}{\mathrm{Re}_D^{1/4}}\right)\rho V^2 = \frac{1}{8}\left(0.316\left(\frac{\rho V D}{\mu}\right)^{-1/4}\right)\rho V^2$$

Transform a circular pipe variables into a flat plate:

$$R = \delta, \ V = 0.817U_0$$

$$\frac{u}{U_0} = \left(\frac{y}{R}\right)^{1/7}$$

$$\bar{V} = \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_0\left(\frac{y}{R}\right)^{1/7} 2\pi r dr$$

$$= \frac{2}{R^2}\int_0^R U_0\left(\frac{R-r}{R}\right)^{1/7} r dr, \ (y = R-r)$$

$$= \frac{2U_0}{R^2}\int_0^R \left(\frac{R-r}{R}\right)^{1/7} r dr$$

$$= U_0 \frac{2\times(1/7)^2}{(1/7+1)(2\times1/7+1)}$$

$$= 0.817U_0$$

Wall shear stress in a flat plate turbulent BL:

$$\tau_w = \frac{1}{8}\left(0.316\left(\frac{\rho(0.817U_0)(2\delta)}{\mu}\right)^{-1/4}\right)\rho(0.817U_0)^2$$

$$= \boxed{0.0233\rho U_0^{\ 2}\left(\frac{\nu}{U_0\delta}\right)^{1/4}}$$ **Blasius's equation**

Application of MIE-Turbulent boundary layer

Local friction factor(C_f) distributions with Reynolds number (p.109 White)

For the smooth circular pipe

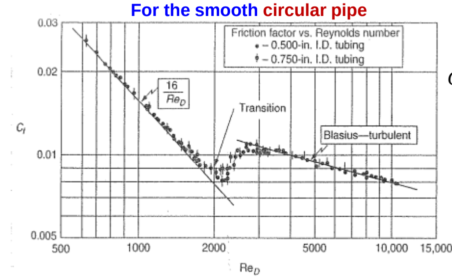


FIGURE 3-7
Comparison of theory and experiment for the friction factor of air flowing in small-bore tubes. [After Sei and Rothfus (1953).]

$$C_f = \frac{f}{4}$$

$$f = \frac{64}{\mathrm{Re}_D} \text{(laminar)}$$

$$= 0.316\,\mathrm{Re}_D^{-\frac{1}{4}} \text{(trubulent)}$$

proposed by Blasius

Application of MIE-Turbulent boundary layer

Combining the momentum Integral eq. and the wall shear stress, we obtain:

$$0.0233\rho U_0^{\ 2}\left(\frac{\nu}{U_0\delta}\right)^{1/4} = \frac{7}{72}\rho U_0^{\ 2}\frac{d\delta}{dx}$$

Separating the variables and then integrating, we obtain:

$$\delta^{1/4}d\delta = 0.240\left(\frac{\nu}{U_0}\right)^{1/4} dx$$

$$\frac{4}{5}\delta^{5/4} = 0.240\left(\frac{\nu}{U_0}\right)^{1/4} x + C$$

$$= 0.240\left(\frac{\nu}{U_0}\right)^{1/4} x, \ (\delta \approx 0 \text{ at } x = 0 \rightarrow C = 0)$$

$$\delta = 0.382\left(\frac{\nu}{U_0}\right)^{1/5} x^{4/5} = 0.382\left(\frac{\nu}{U_0 x}\right)^{1/5} x$$

$$\frac{\delta}{x} = 0.382\left(\frac{\nu}{U_0 x}\right)^{1/5} = \boxed{\frac{0.382}{\mathrm{Re}_x^{\ 1/5}}} \qquad \therefore \delta \sim \mathrm{Re}_x^{-\frac{1}{5}}$$

Blasius's equation

$$\boxed{\tau_w = 0.0233\rho U_0^{\ 2}\left(\frac{\nu}{U_0\delta}\right)^{1/4}}$$

Application of MIE-Turbulent boundary layer

Skin friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^{\ 2}} = 2\times\frac{0.0233\rho U^2}{\rho U_0^{\ 2}}\left(\frac{\nu}{U_0\delta}\right)^{\frac{1}{4}} = 0.0466\left(\frac{\nu}{U_0\delta}\right)^{\frac{1}{4}} = 0.0466\,\mathrm{Re}_\delta^{-\frac{1}{4}}$$

Substituting $\delta = 0.382\left(\frac{\nu}{Ux}\right)^{1/5} x$ **into the equation:**

$$C_f = 0.0466\left(\frac{\nu}{U_0\delta}\right)^{\frac{1}{4}} = 0.0466\left(\frac{\nu}{U_0}\right)^{\frac{1}{4}}\delta^{-\frac{1}{4}} = 0.0466\left(\frac{\nu}{U_0}\right)^{\frac{1}{4}}\left(0.382\left(\frac{\nu}{U_0 x}\right)^{1/5} x\right)^{-\frac{1}{4}}$$

$$= 0.0594\left(\frac{\nu}{U_0}\right)^{\frac{1}{4}}\left(\left(\frac{\nu}{U_0 x}\right)^{-\frac{1}{20}} x^{-\frac{1}{4}}\right) = 0.0594\left(\frac{\nu}{U_0}\right)^{\frac{1}{4}}\left(\frac{\nu}{U_0}\right)^{-\frac{1}{20}}\left(x^{-\frac{4}{20}}\right)$$

$$= 0.0594\left(\frac{\nu}{U_0}\right)^{\frac{4}{20}}\left(x^{-\frac{4}{20}}\right) = 0.0594\left(\frac{\nu}{U_0}\right)^{\frac{1}{5}}\left(x^{-\frac{1}{5}}\right) = 0.0594\left(\frac{\nu}{U_0 x}\right)^{\frac{1}{5}}$$

$$= 0.0594\,\mathrm{Re}_x^{-\frac{1}{5}} = \boxed{\frac{0.0594}{\mathrm{Re}_x^{1/5}}}$$

Application of MIE-Turbulent boundary layer

The overall skin friction coefficient:

$$\overline{C}_f = \frac{1}{L}\int_0^L \frac{0.0594}{\mathrm{Re}_x^{\frac{1}{5}}}dx = \frac{0.0594}{L}\left(\frac{\nu}{U_0}\right)^{\frac{1}{5}}\int_0^L \frac{1}{(x)^{\frac{1}{5}}}dx = \frac{0.0594}{L}\left(\frac{\nu}{U_0}\right)^{\frac{1}{5}}\frac{5}{4}L^{\frac{4}{5}}$$

$$= \frac{0.0743}{\mathrm{Re}^{1/5}}$$

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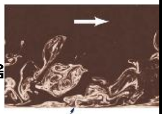
9.2.5 Turbulent BL flow

✓난류 경계층(turbulent boundary layer) 유동

서로 다른 크기의 와도(eddy or vortex)의 비정상 특성 현상을 보임
층류 경계층에 대한 Blasius solution과 같은 엄밀해가 존재하지 않음
난류 경계층의 경우 정확한 전단응력 표현 식이 없으므로 해를 구할 수 없음
따라서 운동량 적분방정식을 이용 엄밀해 대신 근사해를 구함.



✓운동량 적분방정식을 이용한 난류경계층 근사해(예제9.6 참조)

1) 속도분포의 적절한 가정(8장: 난류 유동의 속도분포 참조)
주: 난류유동의 경우 완벽한 속도분포 관계식이 없음
(1) 벽법칙, (2) 속도결손법칙, (3) 1/7승 법칙

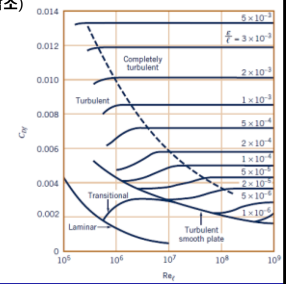
2) 벽면 전단응력(τ_w) 관계식: 경험식

$$\tau_{wall} = 0.025\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}$$

✓평판에서 항력계수 계산: $C_{Df} = f\left(\mathrm{Re}_\ell, \varepsilon/\ell\right)$

Empirical Equations for the Flat Plate Drag Coefficient (Ref. 1)

Equation	Flow Conditions
$C_{Df} = 1.328/(\mathrm{Re}_\ell)^{0.5}$	Laminar flow
$C_{Df} = 0.455/(\log \mathrm{Re}_\ell)^{2.58} - 1700/\mathrm{Re}_\ell$	Transitional with $\mathrm{Re}_{xcr} = 5 \times 10^5$
$C_{Df} = 0.455/(\log \mathrm{Re}_\ell)^{2.58}$	Turbulent, smooth plate
$C_{Df} = [1.89 - 1.62 \log(\varepsilon/\ell)]^{-2.5}$	Completely turbulent



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Boundary Layer
with pressure gradient ($dp/dx\neq0$)

외부유동 (external flow)임에 유의하자

Method	Laminar flow		Turbulent Flow	
	$dp/dx=0$	$dp/dx\neq0$	$dp/dx=0$	$dp/dx\neq0$
Exact Solution (Blasius solution)	O			
Approximate solution (Von Karman MIE)	O	O	O	O

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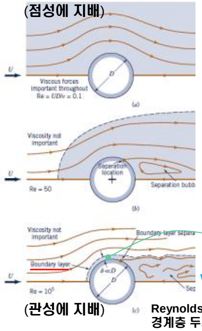
9.2.6 Effects of Pressure Gradient

✓압력구배(pressure gradient)가 경계층(boundary layer) 유동에 미치는 영향

지금까지 평판 위의 압력이 일정한 경계층($dp/dx=0$)에 대한 것을 살펴보았음
경계층에 압력 구배가 있을 경우($dp/dx\neq0$)에 대해 알아보자
예) 평판이 아닌 물체(곡면 등) 주위 유동장에서 압력구배 발생

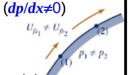
(점성에 지배)

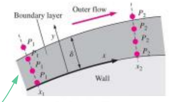


(관성에 지배)

Reynolds No.가 증가하면
경계층 두께가 얇아진다.

높은 Reynolds No.에서 경계층 두께가 얇을 경우
경계층을 가로지르는(y 방향) 압력은 무시할만하고
경계층 안쪽과 바깥쪽의 압력은 서로 유사
경계층을 따르는(x방향) 압력은 변화하며 ($dp/dx\neq0$)
주요 원인은 자유유동속도(U_{fs})의 변화임

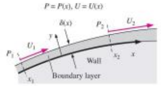




$$\frac{\partial p}{\partial x} = \frac{dp}{dx} \rightarrow p = p(x)$$

$$\frac{p}{\rho} + \frac{1}{2}U^2 = C$$

$$\frac{1}{\rho}\frac{dp}{dx} + U\frac{dU}{dx} = 0 \rightarrow \frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx}$$



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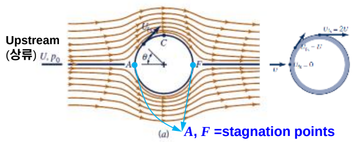
9.2.6 Effects of Pressure Gradient

1) **비점성 유동**에서 압력구배 영향: 경계층 내부 및 외부의 유동특성에 큰 영향을 끼침

예) 직경이 D인 실린더 주위의 유동



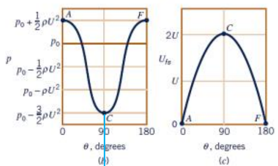
- 비점성유동의 경우: 대칭 유동구조 형성

$$\mathrm{Re} = \frac{\rho V D}{\mu} = \frac{\rho V D}{0} = \infty$$

- 실린더 주위의 속도변화

6.6.3 퍼텐셜 유동 해석 결과

$$U_{fs,A} = U_{fs,F} = 0 \rightarrow U_{fs,c} = 2U$$



- 실린더 주위의 압력변화: 비점성 효과로 에너지 보존

$$p_A = p_F = p_o + \frac{1}{2}\rho U^2 \text{ at } A \text{ and } F \text{ (stagnation point)}$$

$$p_c = \left(p_o + \frac{1}{2}\rho U^2\right) - \frac{4}{2}\rho U^2 = \left(p_o - \frac{3}{2}\rho U^2\right) \text{ at } C$$

실린더 주위의 항력변화

Drag force $= 0 \, (\because \tau_w = 0)$ (압력 분포가 대칭)

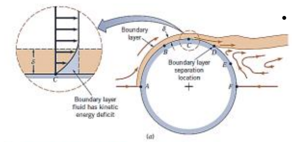
(D'alembert의 역설:
비점성 유체에서 물체의 항력은 0이다.
그러나 실제 항력은 0이 아니다. 왜?)

9.2.6 Effects of Pressure Gradient

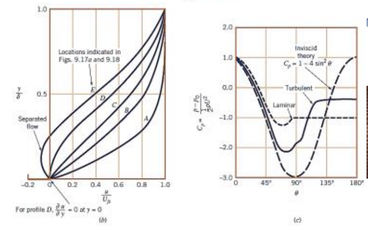
2) **점성 유동**에서 압력구배 영향: 경계층 내부 및 외부의 유동특성에 큰 영향을 끼침

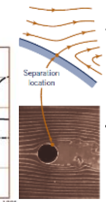
점성 유동장에서는 점성이 아무리 작아도 μ 값에 관계없이 유한한 항력이 측정된다.

→ 주요 원인은 **압력구배**의 영향



- 점성유동의 경우:
  - 점성 마찰로 경계층 내 운동 에너지 손실 발생
  - 에너지 공급이 없으므로 유동박리(separation)발생
  - 박리점에서 속도=0, τ_w=0
  - 박리점 이후에서 역류(reverse flow) 발생





- 실린더 주위의 압력변화

$p_A = p_F$ for inviscid flow

$p_A \gg p_F$ for viscous flow

- 실린더 주위의 항력변화

Drag force $\neq 0 \, (\because \tau_w \neq 0)$

점성마찰 항력은 거의 0

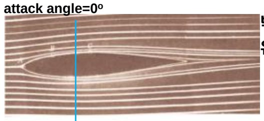
그러나 압력에 의한 항력 발생

(D'alembert의 역설이 설명됨)

9.2.6 Effects of Pressure Gradient

예) 영각(attack angle)을 갖는 날개 주위 유동

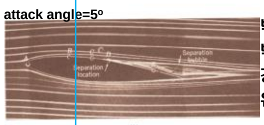
attack angle=0°



박리가 거의 관찰되지 않음

유선형 물체: 박리 영향이 제거 또는 줄이도록 설계된 것

attack angle=5°



박리가 D 점에서 관찰되고 있음

박리영역(후류)에서 낮은 압력으로 압력항력이 증가한다

경계층이 아무리 얇더라도 박리 발생은

유동장을 상당히 변화시킨다.

$\frac{\partial p}{\partial x} < 0$ 순압력구배 | $\frac{\partial p}{\partial x} > 0$ 역압력구배

9.3 Drag (항력)

9.3 Drag (항력)

✓ Classification of drag force (항력)

Drag force (전항력)

형상항력 (Profile drag)

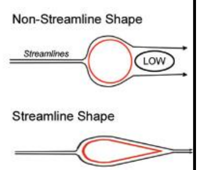
- Pressure drag(압력항력) / Form drag (형체항력): 유동박리로 인해 물체 전후방 압력차에 의한 항력. 후류(wake)의 크기, 박리점 위치에 따라 변화
- Skin drag (표면항력) / Friction drag(마찰항력): 유체 점성으로 물체 표면에 작용하는 전단력의 총합. 층류의 경우 대수계산 가능. 난류의 경우 복잡

※유선형물체(streamlined body): 유동박리가 일어나지 않아 압력항력이 줄어들도록 만든 물체. 즉, 전항력이 표면마찰에 의해 발생

$$\text{Drag force}: F_D = D_{drag} = C_d \frac{1}{2}\rho U_0^2 A$$

$$\text{Drag coefficient}: C_d = \frac{F_D}{\frac{1}{2}\rho U_0^2 A}$$

$$: C_d = \frac{F_D}{\frac{1}{2}\rho U_0^2 2A} \text{ (submerged body)}$$



C_d is as a function of Reynolds No., Mach No., Froude No., Relative roughness

$$C_D = \phi(\text{shape, Re, Ma, Fr, } \varepsilon/\ell)$$

9.3.1 Frictional Drag (마찰항력)

✓마찰 항력(friction drag , D_f)

물체표면의 전단응력(τ_w)에 의해 발생하는 항력. Reynolds No.가 낮은 경우 지배적

$$D_f = C_{Df} \frac{1}{2}\rho U^2 \underbrace{b\ell}_{A} \rightarrow C_{Df} = \frac{D_f}{\frac{1}{2}\rho U^2 b\ell}$$

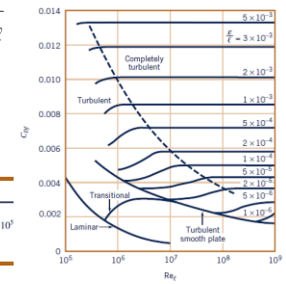
마찰 항력계수(friction drag coefficient)

$$C_{Df} = f(\text{Re}_\ell, \varepsilon/\ell)$$

평판 유동의 경우 마찰항력계수

Empirical Equations for the Flat Plate Drag Coefficient (Ref. 1)

Equation	Flow Conditions
$C_{Df} = 1.328/(\text{Re}_\ell)^{0.5}$	Laminar flow
$C_{Df} = 0.455/(\log \text{Re}_\ell)^{2.58} - 1700/\text{Re}_\ell$	Transitional with $\text{Re}_{xcr} = 5 \times 10^5$
$C_{Df} = 0.455/(\log \text{Re}_\ell)^{2.58}$	Turbulent, smooth plate
$C_{Df} = [1.89 - 1.62 \log(\varepsilon/\ell)]^{-2.5}$	Completely turbulent



곡면 유동의 경우 마찰항력계수: 구하기가 매우 어려움(실험적 결과 사용)

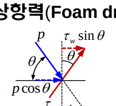
9.3.2 Pressure Drag (압력항력)

✓압력 항력(pressure drag, D_p)

유동박리에 의한 물체 전후에 발생하는 압력차에 의한 항력.

물체의 형상과 밀접한 관계가 있으므로 형상항력(Foam drag)라고도 함

$$D_p = \int p\cos\theta dA$$

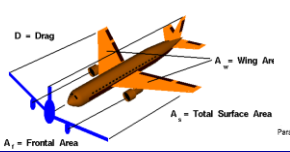


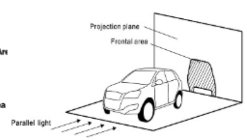
압력 항력계수(pressure drag coefficient)

$$C_{Dp} = \frac{D_p}{\frac{1}{2}\rho U^2 A} = \frac{\int p\cos\theta dA}{\frac{1}{2}\rho U^2 A} = \frac{\int C_p \cos\theta dA}{A}, \quad C_p = \frac{(p - p_o)}{\frac{1}{2}\rho U^2}$$

기준압력 (대기 중: 대기압)

(pressure coefficient)





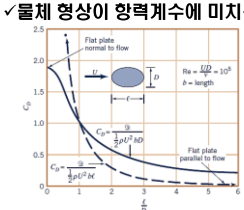
9.3.3 Drag Coefficient and Example

압력 항력계수(pressure drag coefficient)에 영향을 미치는 변수

C_D is as a function of Reynolds No., Mach No., Froude No., Relative roughness

$$C_D = \phi(\text{shape, Re, Ma, Fr, } \varepsilon/\ell)$$

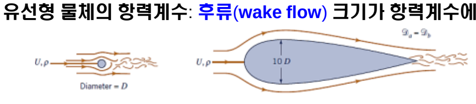
✓물체 형상이 항력계수에 미치는 영향(Shape dependence)



$$A = bD \text{ or } b\ell$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 bD} \text{ or } \frac{D}{\frac{1}{2}\rho U^2 b\ell}$$

유선형 물체의 항력계수: 후류(wake flow) 크기가 항력계수에 지배적 영향



$C_D = 1.2$ $C_D = 1.2$

9.3.3 Drag Coefficient and Example

✓Reynolds No.가 항력계수에 미치는 영향 (Reynolds Number dependence)

1) Low Reynolds No. 유동(Re < 1): 점성력과 압력힘이 균형(관성력은 무시할만함)

$$D = f(U, \ell, \mu)$$

$$D = C\mu\ell U$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \ell^2} = \frac{2C\mu\ell U}{\rho U^2 \ell^2} = \frac{2C\mu}{\rho U \ell} = \frac{2C}{\mathrm{Re}}$$ (항력계수는 Re에 반비례)

Low Reynolds Number Drag Coefficients (Ref. 7) ($\mathrm{Re} = \rho UD/\mu$, $A = \pi D^2/4$)

Object	$C_D = \mathscr{D}/(\rho U^2 A/2)$ (for Re $\lesssim$ 1)	Object	C_D
a. Circular disk normal to flow	20.4/Re	c. Sphere	24.0/Re
b. Circular disk parallel to flow	13.6/Re	d. Hemisphere	22.2/Re

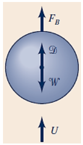
Example

Example 9.10 Low Reynolds Number Flow Drag (Stokes Law: 중요)

GIVEN: While workers spray paint the ceiling of a room, numerous small paint aerosols are dispersed into the air. Eventually these particles will settle out and fall to the floor or other surfaces. Consider a small spherical paint particle of diameter D=1x10^{-5} m (or 10 μm) and specific gravity SG =1.2.

FIND: Determine the time it would take this particle to fall 2.4 m from near the ceiling to the floor. Assume that the air within the room is motionless.

SOLUTION



$$W_{weight} = D_{drag} + F_{bouyant}$$

$$W = \gamma_{paint}\forall = S\gamma_{H2O}\frac{\pi}{6}D^3,\quad F_B = \gamma_{air}\forall = \gamma_{air}\frac{\pi}{6}D^3$$

$$D = C_D\frac{1}{2}\rho_{air}U^2\frac{\pi}{4}D^2 = \frac{1}{2}\rho_{air}U^2\frac{\pi}{4}D^2\left(\frac{24}{\rho_{air}UD/\mu_{air}}\right) \rightarrow D = 3\pi\mu_{air}UD$$

(from Table 9.4)

$$S\gamma_{H2O}\frac{\pi}{6}D^3 = 3\pi\mu_{air}UD + \gamma_{air}\frac{\pi}{6}D^3$$

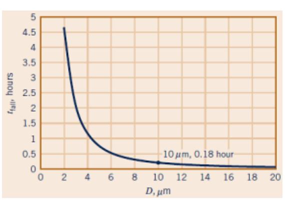
$$U = \frac{D^2(S\gamma_{H2O} - \gamma_{air})}{18\mu_{air}} = \frac{(10^{-5}m)^2\left[(1.2)(9800N/m^3) - 12.0N/m^3\right]}{18(1.79\times10^{-5}Ns/m^2)}$$

$$= 0.00365m/s$$

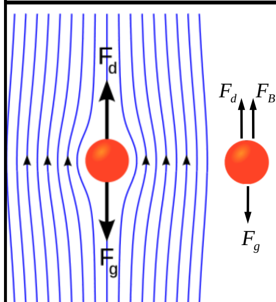
Example

$$t_{fall} = \frac{2.44m}{0.00365m/s} = 668s$$

$$\mathrm{Re} = \frac{\rho_{air}UD}{\mu_{air}} = \frac{1.23kg/m^3\left(1\times10^{-5}m\right)0.00365m/s}{1.79\times10^{-5}Ns/m^2} = 0.00251 << 1$$



S. Stokes flow & Stokes' law



✓Assumptions

① low Reynolds number (Re<<1)
② No flow separation around sphere
③ Wake flow is laminar
④ No acceleration (No body force)
⑤ friction drag is dominant (due to no separation)

✓Drag force (항력): Reynolds No.의 함수
물체의 운동방향과 반대방향으로 작용하는 저항력

Drag force
- F_{ST}: Friction drag force (Re < 1)
  유체 점성의 영향으로 발생, τ_w의 적분값
- F_{SN}: Pressure drag force (Re>1000)
  유동박리에 의한 물체 전후 압력차에 기인

An object moving through a gas or liquid experiences a force in direction opposite to its motion.
Terminal velocity is achieved when the drag force is equal in magnitude but opposite in direction to the force propelling the object.

S. Stokes flow & Stokes' law

✓**Stokes' law (Stokes' drag force law)**

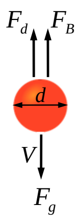
$$\boxed{\text{Drag force } F_d = 3\pi\mu V d}$$

- F_{ST} : Friction drag $F_d = 2\pi\mu V d$
- F_{SN} : Pressure drag $F_d = 1\pi\mu V d$

Derived by Stokes over 160 years ago! (1851)
Two thirds of the force is due to viscous shear
One thirds of the force is due to pressure drag

앞장의 가정, N-S eq.**을 이용 해석적으로 얻은 해 유도과정은 너무 복잡하여 생략**

✓**Forces acting on a sphere moving fluids (gas or liquid)**



Drag force $F_d = 3\pi\mu V d$

Bouant force $F_B = \frac{\pi}{6} d^3 \rho_m g$

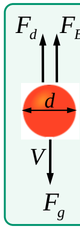
Gravitational force $F_g = \frac{\pi}{6} d^3 \rho_p g$

V : particle velocity
d : particle diameter
ρ_p : particle density
ρ_m : density of medium
μ : viscosity of medium

S. Stokes flow & Stokes' law

✓**Drag force of sphere in fluid (Stokes' drag force law)**

Drag coefficient (항력계수) is a dimensionless quantity that is used to quantify the drag or resistance of an object in a fluid. The drag coefficient is always associated with a particular surface area, and it comprises the effects of the two basic contributors to fluid dynamic drag:
skin friction(friction drag) and form drag(pressure drag).



$$C_d = \frac{F_d}{\frac{1}{2}\rho V^2 A}$$

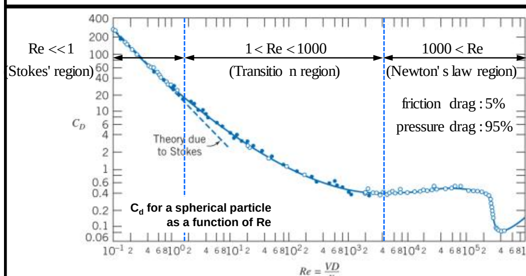
$$F_d = 3\pi\mu V d$$

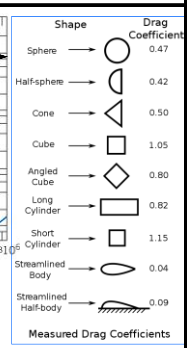
$$C_d = \frac{F_d}{\frac{1}{2}\rho V^2 A} = \frac{3\pi\mu V d}{\frac{1}{2}\rho V^2 \frac{\pi d^2}{4}} = \frac{24\mu}{\rho u d}$$

$$\boxed{C_d = \frac{24}{\text{Re}} \quad (\text{for Re} << 1 : \text{stokes' law})}$$

C_d varies as a function of speed, flow direction, object shape, object size, fluid density and fluid viscosity. Speed, kinematic viscosity and a characteristic length scale of the object are incorporated into a dimensionless quantity called the Reynolds number.

S. Stokes flow & Stokes' law





$C_d = \frac{24}{\text{Re}}$ Re << 1 (Stokes' law)

$C_d = \frac{24}{\text{Re}}\left[1 + \frac{3}{16}\text{Re} + \frac{9}{160}\text{Re}^2 \ln \text{Re} + \cdots\right]$ 1 < Re < 2 (Oseen eq.)

$C_d = \frac{24}{\text{Re}}\left[1 + 0.15\,\text{Re}^{0.678}\right]$ 2 < Re < 500

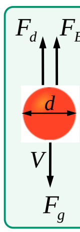
$C_d \approx 0.44$ 500 < Re < 2×10^5 (Newton's law)

$C_d = 1 + 10\,\text{Re}^{-\frac{2}{3}}$ (for cylinder) 1 < Re < 2×10^5

S. Stokes flow & Stokes' law

✓**Terminal velocity (낙하하는 구의 종말속도, $V_{terminal}$)**

As the object, the drag force acting on the object increases. At a particular speed, the drag force produced will equal the object's weight (*mg*). At this point the object ceases to accelerate altogether and continues falling at a constant speed called terminal velocity (also called settling velocity).



$$C_d = \frac{F_d}{\frac{1}{2}\rho V^2 A}$$

$$\sum F_z = W - F_d - F_B = 0 \;(\text{No acceration})$$

$$= \frac{\pi}{6} d^3 \rho_p g - 3\pi\mu V d - \frac{\pi}{6} d^3 \rho_m g$$

$$3\pi\mu V d = \frac{\pi}{6} d^3 (\rho_p - \rho_m) g$$

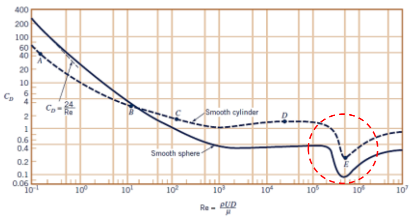
$$V_{\text{terminal}} = \frac{d^2(\rho_p - \rho_m) g}{18\mu}$$

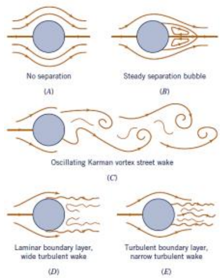
예를 들어 1g**의 빗방울이** 10km **상공에서 떨어지면 엄청난 가속도가 붙지만 우리는 빗방울에 맞아 죽지 않는다. 이것이** Stokes **법칙이다.**

9.3.3 Drag Coefficient and Example

2) Intermediate Reynolds No. 유동($10^3 < Re < 10^5$):

경계층 구조를 가짐. Re증가에 따라 C_d 약간 감소

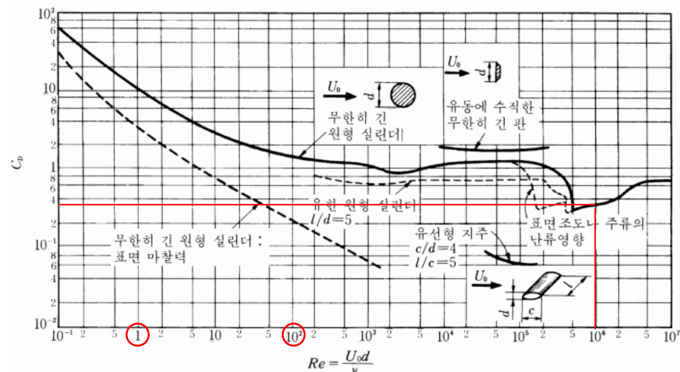




층류 경계층에서 난류경계층으로 천이 영역: 항력이 감소(난류에 의한 후류 영역의 크기가 감소

$$Re_{x,cr} \approx 2\times10^5 \sim 3\times10^6 \;\rightarrow\; Re_x = 5\times10^5 \quad \text{for flat plate}$$

9.3.3 Drag Coefficient and Example



마찰저항이 전체 2/3

압력저항이 전체 90% 이상

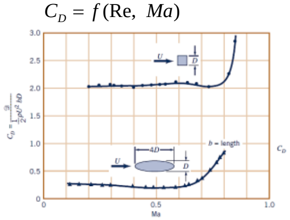
9.3.3 Drag Coefficient and Example

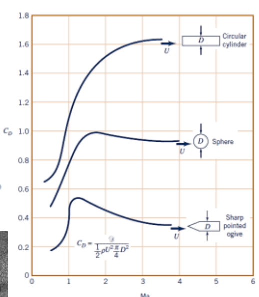
✓압축성 영향 (Compressible effect): Mach No.와 Re. No.의 함수

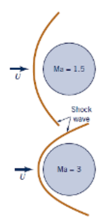
Mach No.(마하수) 0.5이하에서는 압축성 영향이 거의 무시됨

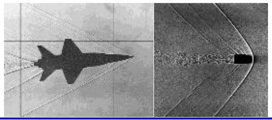
Mach No.(마하수) 1.0 근처에서 항력계수가 갑자기 증가: 충격파(shock wave)의 영향

$$C_D = f(\mathrm{Re},\; Ma)$$









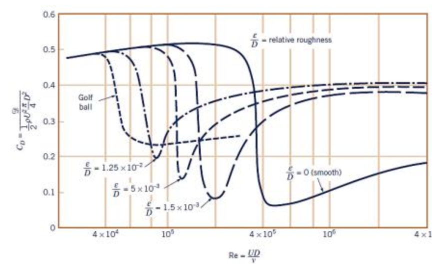
9.3.3 Drag Coefficient and Example

✓표면 거칠기 영향(Surface roughness)

난류 경계층에서 지배적 영향이 나타남.

표면 거칠기가 증가하면 벽면전단응력이 증가하며 층류에서 난류로의 천이 구간이 변함

따라서 Re. No.에 따라 항력은 증가할 수도 감소할 수도 있음



$Re_c = 4\times10^4$

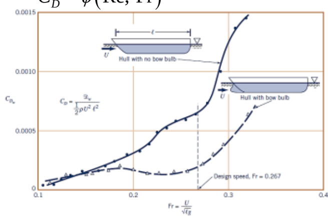
$Re_c < 4\times10^4$

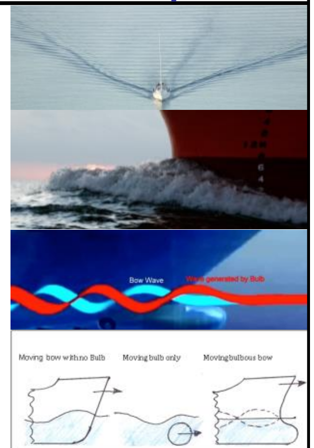
9.3.3 Drag Coefficient and Example

✓**Froude Number effect**

$$\mathrm{Fr} = \frac{U}{\sqrt{g\ell}}$$

$$C_D = \phi(\mathrm{Re},\ \mathrm{Fr})$$



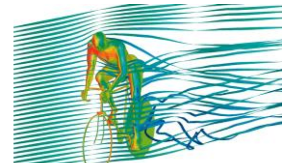


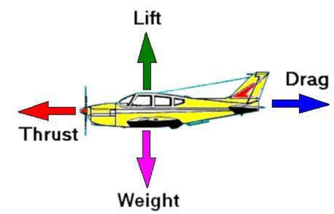
✓**복합물체의 항력(Composite Body Drag)**

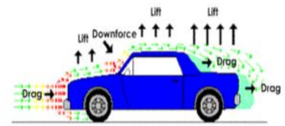
단순화된 물체의 항력을 구한 후 총합을 구한다

(이 방법은 주의를 해야 함)

9.4 Lift (양력)



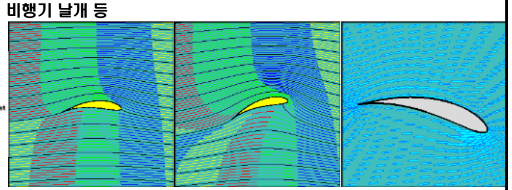




9.4 Lift (양력)

✓**양력(Lifting force): 비대칭 물체 또는 회전하는 구 주위에 발생하는 공기역학적 힘**

양력을 증가시키는 물체: 비행기 날개 등



양력을 감소시키는 물체: 자동차 스포일러(downforce wing)

9.4.1 Surface Pressure Distribution

✓**물체 표면 압력 분포**

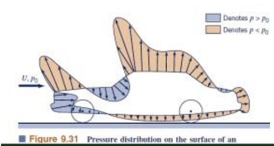
양력은 점성력 보다 물체 표면 압력분포로부터 발생

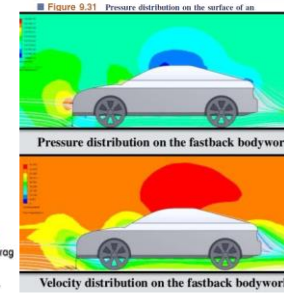
압력분포의 적분값을 구하는 것이 쉽지 않음:

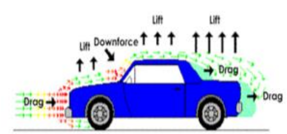
무차원 양력계수 사용

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

$$C_L = \phi(\mathrm{shape},\ \mathrm{Re},\ \mathrm{Ma},\ \mathrm{Fr},\ \varepsilon/\ell)$$





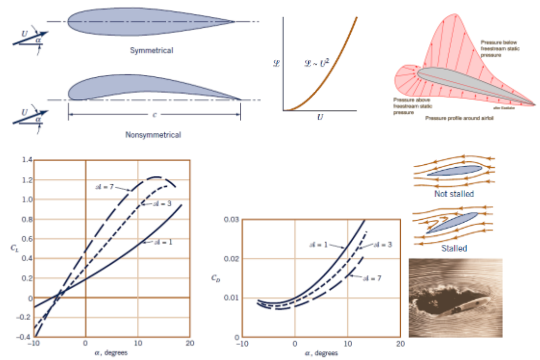


9.4.1 Surface Pressure Distribution

✓익형(airfoil)에서 양력 발생

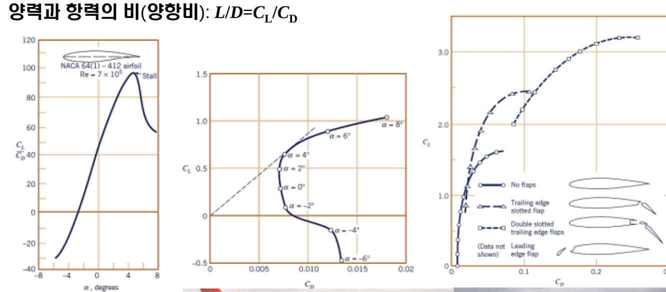
Reynolds No.가 큰 유동에서 압력분포는 동압에 비례. 점성효과 무시할 정도

양력은 공기속도의 제곱에 비례



9.4.1 Surface Pressure Distribution

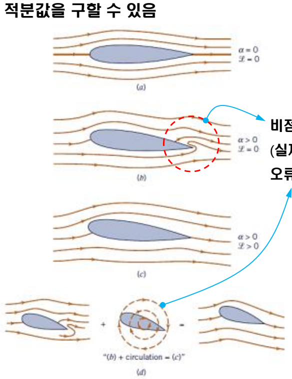
양력과 항력의 비(양항비): $L/D=C_L/C_D$



9.4.2 Circulation (순환)

✓순환(circulation):

양력생성에 있어 점성이 중요한 역할을 하지 않으므로 비점성유동 방정식으로부터 압력분포의 적분값을 구할 수 있음



비점성유동 방정식으로부터 얻은 오류

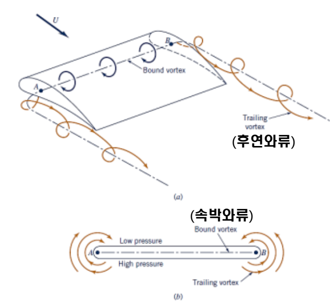
(실제유동과 차이가 있음)

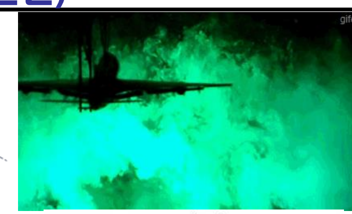
오류수정방법: 익형 주위에 적절한 시계방향 선회유동을 더한다

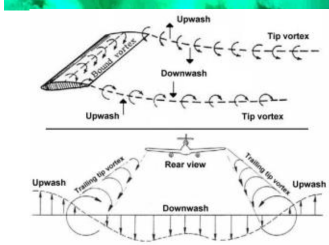
(순환의 합성)

9.4.2 Circulation (순환)

✓익형 주위의 와류



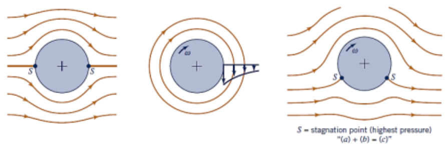


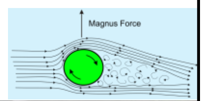


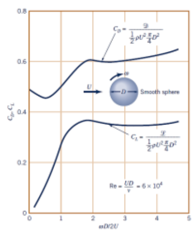
후연와류+속박와류= 말굽와류

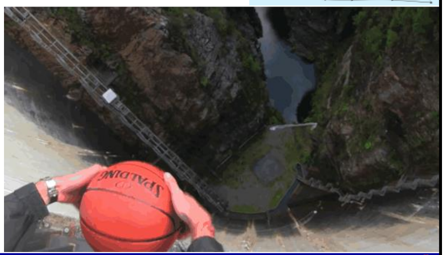
9.4.2 Circulation (순환)

✓Magnus 효과









Homework #4

✓Problem set for Ch. 9

2, 8, 12, 20, 26,
28, 36, 58, 72,74,
84, 102, 104, 114
(14 problems)

제출기한: **chapter** 9. **시험 보기 직전**